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UNIVERSITY COLLEGE, UNIVERSITY OF LONDON

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TECHNICAL SERIES, V.

AN EXPERIMENTAL STUDY OF THE STRESSES IN MASONRY DAMS.

BY

KARL PEARSON, F.R.S., AND A. F. CAMPBELL POLLARD,

ASSISTED BY

C. W. WHEEN AND L. F. RICHARDSON.

[WITH NINE FIGURES IN THE TEXT AND ELEVEN PLATES.]

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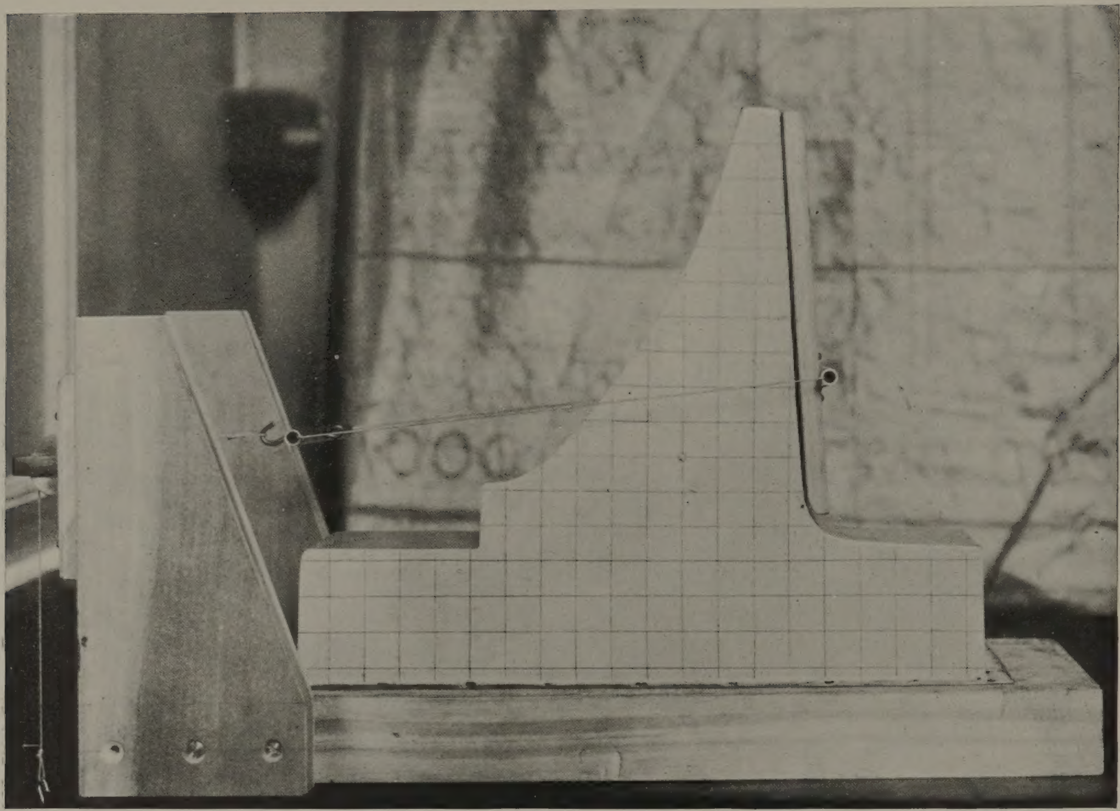

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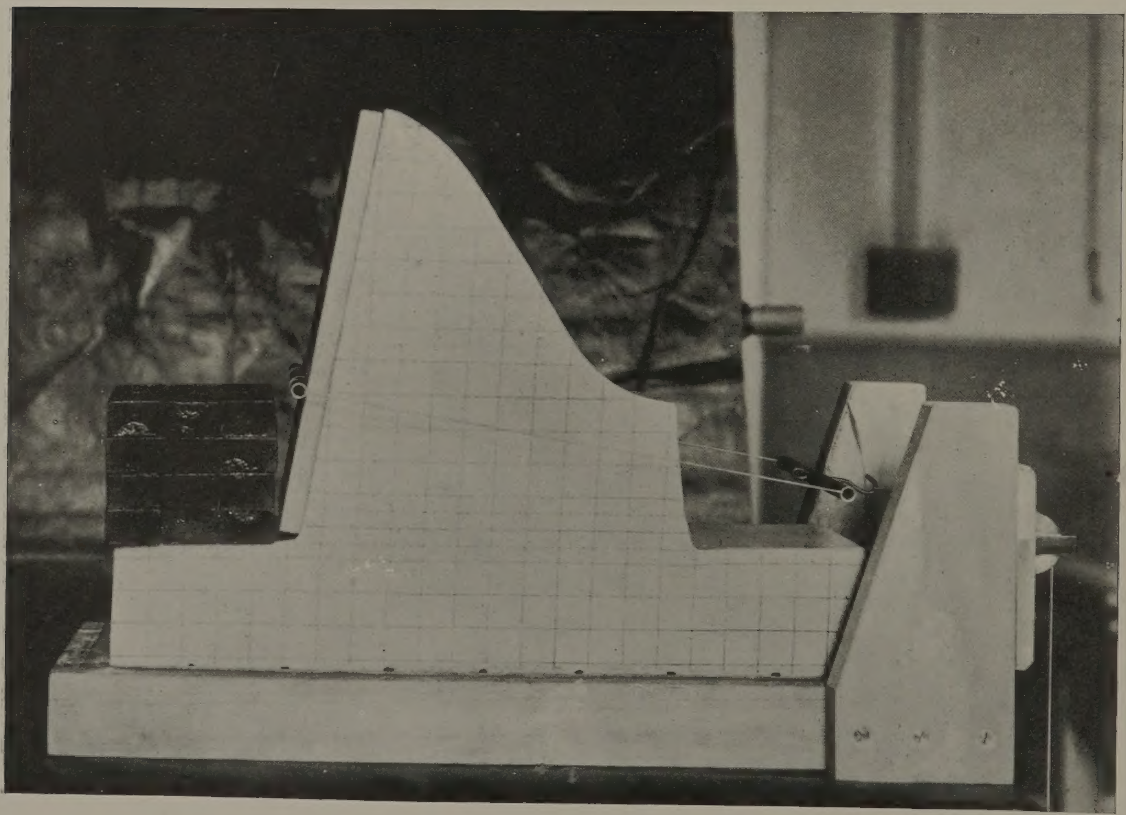

Overloaded Vyrnwy-Type Dam.

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An Experimental Study of the Stresses in Masonry Dams.

By Karl Pearson, F.R.S., and A. F. Campbell Pollard, assisted by
C. W. Wheen and L. F. Richardson.

(1) *Introductory.* The present report on the work on model dams conducted during the last three years in the Department of Applied Mathematics at University College, London, contains, as must frequently be the case in like researches, an account of scarcely a tithe of the investigations, experimental and theoretical, which have been carried out or attempted. Many weeks, it would not be exaggeration to say many months, have been spent in the endeavour to obtain not exact, but approximate solutions of Equation (iii) suited (a) to current types of dam contour, and (b) to what we have ascertained experimentally to be the general form of shear distribution over the base. If we have been unsuccessful hitherto in these attempts, it does not follow that success may not later attend our or probably others' efforts. The real difficulties of the problem centre about two points, (i) the fact that the breadth of the dam being of the same order as its height, there is no small quantity in terms of which the mathematician can expand his solution and so obtain an approximate series*, (ii) our ignorance of the distribution of shear over the base

* Since the paper by Atcherley and Pearson was published, a memoir on dams by Mr Max am Ende has been issued in *Engineering*, Dec. 8, 1905. The author speaks as if his memoir were a solution of the problem. He treats the dam as a cantilever, and obtains such familiar results as the linearity of the normal stresses perpendicular to a cross-section, and the parabolic character of the shearing stresses. Now his fundamental equations are not consistent with the theory of elasticity, they apply only to the cases of beam or arch, where (i) the dimensions of the cross-section are small compared to the length, (ii) where those dimensions are small compared to the radius of curvature of the central line, and (iii) where true cross-sections, i.e. sections *perpendicular* to the line of their centroids exist. Not one of these conditions is fulfilled in the case of masonry dams (as distinguished from arches and buttresses), and the whole problem turns on how far the results obtained by assuming them are consonant with experience. Experiment shows that they are not even approximately fulfilled. Further there are numerous inconsistencies in Mr Max am Ende's treatment, e.g. he takes his XX as the dam's centre line without further definition, apparently it is the line of centroids of horizontal sections, but afterwards it appears to be something quite different. It is not an easy problem to find for a dam a series of cross-sections perpendicular to their line of centroids. A knowledge of the general theory of elasticity would have shown the writer that the formula cited from Winkler is wholly inapplicable. Further the elastic constants of a dam do not directly involve the stretch modulus at all (see our Equations pp. 11—12), and a test of the stability depending not on strains, but stresses, is fallacious. Throughout Mr Max am Ende does not recognise that what we may term beam, and he terms cantilever formulae, depend for their

of the structure. Experimental work gives some slight clue as to the nature of this distribution in dams of various types, but it also shows that the character of it is immensely modified by the exact "fixing" of the substratum and by the form of the dam at its tail. After much expenditure of energy we have been forced to the conclusion that at present at any rate the only solution we can offer is the experimental one, i.e. determine by a model of the proposed dam the shear distributions over the sections and then calculate the strains in the actual dam from the observed form of the distributions. This method is exemplified in the following memoir in a dam of Assuan type. It is perfectly true that the method is lengthy and difficult and the results when obtained only approximate. But in dealing with a structure of this kind the expense of such an investigation is a very small element of the ultimate cost, and we have a reasonable certainty—which the old theory does not provide—of not being 10 or 20 per cent. out in our maximum strains.

In publishing our results, however, we wish to state that we consider them only preliminary. The expense attending even experiments on our small scale has been relatively large, and would have been impossible but for the grant made annually for research to this Department of our College by the Worshipful Company of Drapers. We feel convinced that more uniform results would be obtained if much larger model dams, say $6' \times 5'$, were used, and special experiments made to determine the influence of extent and form of substratum on the fundamental shear distribution. We feel convinced that more comprehensive experiments on a much larger scale ought to be undertaken by some public body like the Institution of Civil Engineers.

It cannot be forgotten that this problem of the dam is one of the most important practical questions which presents itself to the civil engineer. Even if existing dams are all definitely stable, the cost of construction is very great and it by no means follows that the amount of material employed is necessary or, if necessary, has been disposed of to the greatest advantage. The theory of dam construction has remained stationary for many years, and this is not satisfactory from either the theoretical or the practical standpoint. But to state that the problem is in this condition is neither to make grave charges against the engineer, nor against the theoretical elastician; it is merely to recognise the great difficulties of the problem and to encourage attempts at its solution. Those attempts must centre round an experimental knowledge of the shearing stress distribution over the cross-sections. The fundamental conception in the existing practical theory is summed up in the "rule of the middle-third" for the horizontal sections. The so-called "line of resistance" is to pass through the middle-third of the horizontal sections. Secondary conditions as to the "angle of friction"

exactness on "the principle of the elastic equivalence of statically equipollent loads" (see Todhunter and Pearson, *History of the Theory of Elasticity and the Strength of Materials*, Vol. II. 8, 9, 21, 100, 1441, 1521, etc.), and this principle has no application when the load is distributed over an area whose linear dimensions are of the same order, as the distance from this area of the point at which the stresses are being considered.

and the safe compressive stress of masonry are of course involved, but these can only be applied after the line of resistance has been constructed and the middle-third condition has been applied and found to be fulfilled*. Now the whole of this middle-third rule is based upon the assumption that the normal stress on a horizontal section of the dam is a linear function of the distance of the point at which we are considering the stress from a horizontal line drawn in the section parallel to the length of the dam. We shall speak of this assumption as the "hypothesis of linear normal stress." The rule of the middle-third as applied to the horizontal sections of a dam stands or falls with this hypothesis of linear normal stress. But it follows at once from the general equations of elasticity that if the normal stress be linear the shearing stress over the horizontal sections will be a quadratic function of the distance from the horizontal line considered above, in other words the distribution of shearing stress will be parabolic. This parabolic distribution was first indicated by Lévy and has been fully discussed in the memoir by Mr Atcherley and one of the present writers†. In that memoir it is shown that the axis of the parabola must be the vertical bisector of the base whenever the tail of the dam is truncated and the front starts with a vertical step. When there is a batter on the front and on the truncated face of the tail, the parabola will be somewhat shifted from its central position. In a dam of the Assuan type, however, worked out in detail in the memoir referred to, it is shown that the parabola is only shifted one foot in the total base length of 73 to 74 feet. But a serious consequence follows from these steps in developing the original hypothesis of linearity of normal stress: The distribution of shear being known, the stresses on the vertical sections can now be calculated, and these will be found, at any rate in some cases, to give tensions and not pressures—in other words the line of resistance for the vertical sections falls outside the middle-third of the vertical sections. Thus we find that the fundamental condition for stability which has been adopted as the criterion for safety, i.e. no tension in the structure, fails if applied consistently all round. This is the criticism of the existing theory presented in the memoir referred to, and we may sum up the state of affairs as far as practical theory is concerned as follows:

(i) The dam shall not be subjected to tensile stresses.

(ii) This involves the line of resistance lying in the middle-third of the horizontal sections.

(iii) This condition (ii) has meaning solely on the assumption that the normal stresses are linear.

(iv) Linearity of the normal stresses involves the distribution of shearing stress being parabolic.

* See Atcherley and Pearson, *On some Disregarded Points in the Stability of Masonry Dams*, Drapers' Company Research Memoirs, Techical Series II, Dulau & Co., p. 6.

† Atcherley and Pearson, *loc. cit.* pp. 23—7.

(v) But a knowledge of the distribution of shearing stress enables us to determine the normal stresses on the vertical sections. If we assume (iii) to be true for the vertical sections, then there will be as a rule tension in the tail of dams. If we do not assume (iii), then we have a more lengthy and laborious determination of the vertical stresses, but until it has been made we have no right to treat the fundamental condition (i) as satisfied because we have shown (ii) holds. The vertical sections may after all be the critical sections.

The object of the present paper is to ascertain what experimental evidence can be brought to bear on the relations (i) to (iv) above.

But before this is done it is as well to consider more at length the mathematics of the simplest case.

(2) *State of Theory.* Let us suppose an indefinitely long dam, or if it be of finite length, that its terminals abut against rigid supports. Suppose the dam to be straight, and its front to be plane but at any batter; its flank may be curved and of any form. Let us limit ourselves to the case of homogeneous isotropic material, representing the actually amorphic mixture of concrete and stones of which dams are usually built up, in an average manner by the dilatation coefficient λ and the slide modulus μ. Then with the notation of Todhunter and Pearson's *History of Elasticity*, if z be the vertical, y the length and x the breadth axis, we have:

$$\left.\begin{aligned} \frac{d\widehat{xx}}{dx} + \frac{d\widehat{xz}}{dz} &= 0 \\ \frac{d\widehat{xz}}{dx} + \frac{d\widehat{zz}}{dz} + \rho g &= 0 \end{aligned}\right\} \qquad \text{(i)},$$

where ρ = the density of the masonry.

The solution of these equations is given by*:

$$\left.\begin{aligned} \widehat{zz} &= \frac{d^2 V}{dx^2} - g\rho z \\ \widehat{xx} &= \frac{d^2 V}{dz^2} \\ \widehat{xz} &= -\frac{d^2 V}{dx\,dz} \end{aligned}\right\} \qquad \text{(ii)},$$

where

$$\left(\frac{d^2}{dx^2} + \frac{d^2}{dz^2}\right)^2 V = 0 \qquad \text{(iii)}.$$

Here the form of V must be selected so as to satisfy the three conditions:

(a) The shearing stress = 0, and the normal stress equals the water pressure over the front of the dam.

* Atcherley and Pearson, *loc. cit.* p. 8, where $2\lambda + 2\mu$ should be read throughout for $\lambda + 2\mu$.

(*b*) The shearing stress $=0$, and the normal stress equals nothing over the top and flank of the dam.

(*c*) The shearing stress and the normal stress have their actual values over the base of the dam.

There are several points to be noted here:

First the condition (*c*) is often forgotten, but its necessity is the real condition which has prevented the mathematician getting further with the problem. The engineer using the middle-third rule and thus assuming the hypothesis of linear normal stress has actually assumed the stresses over the base*. Consciously or unconsciously he has asserted that the pressure is linear and the shear parabolic. For suppose $\widehat{zz}$ linear or

$$\widehat{zz} = f_0(z) + f_1(z)\,x \qquad \text{(iv)}.$$

Then the second equation of (i) gives us:

$$\widehat{xz} = \chi_1'(z) - \{\rho g + f_0'(z)\}\,x - f_1'(z)\,\frac{x^2}{2} \qquad \text{(v)}.$$

This is the demonstration of the parabolic distribution of shear over the horizontal sections.

Further, from the first equation of (i):

$$\widehat{xx} = \chi_0''(z) - \chi_1''(z)\,x + f_0''(z)\,\frac{x^2}{2} + f_1''(z)\,\frac{x^3}{6} \qquad \text{(vi)}.$$

This shows that the normal stress on the vertical sections is fully determined. What its character will be depends on the values at present undetermined of χ_0, χ_1, f_0, f_1, which are arbitrary functions of z. It will be found that

$$V = \chi_0(z) - \chi_1(z)\,x + \frac{x^2}{2}\{g\rho z + f_0(z)\} + \frac{x^3}{6}\{f_1(z)\} \qquad \text{(vii)}.$$

Thus (iv) and (v) show that to assume the normal stress over a horizontal section to be linear is to assume that the stress over the base of the dam is given by a linear pressure and a parabolic shear. On the theory of elasticity these may be anything whatever, notwithstanding that we retain the surface and body loads on the material the same. Clearly in any actual dam the stresses over the base† will depend upon the nature of the substratum and the manner in which the dam itself has been built up. The only justification therefore for the assumption that the pressures are linear over the base and the shears parabolic must depend then on *experimental* evidence, that in models approaching even approximately to a real dam there is approximately a linear pressure or a parabolic shear. To this point we return later.

* This assumption is tacitly made in M. Lévy's solution for a triangular contour dam and can only be justified by experiment.

† Throughout this memoir the term "base of dam" is used for any arbitrary plane drawn horizontally to cut off the dam at substratum level.

Another point now arises. Supposing we accept (iv) to (vi), can we satisfy by means of them, the boundary conditions at the front, top and flank of the dam? The answer is in the negative for every dam we have yet tried, except that of triangular contour. They do not suffice even for the case of a dam with vertical front and flank, still less for a trapezoidal dam,—always supposing, what is actually true, that the breadth of the dam is sensible as compared with its height. Even in the case of a triangular dam the solution is not an exact solution, for we have started from the hypothesis of a definite base distribution, which must be in some way justified by experiment.

Suppose we attempt to modify (iv) by the addition of a function $F(x, z)$, which destroys the linearity of the normal pressure on the horizontal sections. Let any term of this be $\beta x^s z^r$, where if this corrective term be small β may be considered small. Then the second equation of (i) shows us that we introduce into the shear a term

$$-\frac{r}{(s+1)}\frac{x}{z}\beta x^s z^r,$$

or, since x and z are of the same order of magnitude in actual dams* we introduce a term of the same order into the shear. In other words the deviation of the normal stress on the horizontal sections from linearity is of exactly the same order as the deviation of the shear on the same sections from a parabolic distribution. Neither distribution is correct in the case of actual dams, but we have no right whatever to adopt one and discard the other in working for an approximate solution. In the case of several engineering discussions of this subject, it has been supposed that there was something especially probable about the linear distribution of normal stress which did not apply to the parabolic distribution of shear. We see at once that they stand or fall together†.

(3) *Suggested Course of Investigation.* Looking at the matter from the practical side the following problems suggest themselves:

(i) No elastic theory solution is possible until we have some idea of the distribution of pressural and shearing stresses over a dam base. It seems only possible, however, to get rough experimental conceptions of what such distributions may be.

(ii) Even if the base distributions such as arise in practice were accurately known, it would be probably quite impossible for the mathematician to work out the actual stresses in such dams as occur in practice. All he could do, would be to solve perhaps the case of a triangular contour and approximate to the

* In the case of a girder, of course, the order of the coordinate x in the plane of the cross-section may be very small as compared with z, and thus the 'corrective terms' need not be of the same order for tensile and shearing stresses. This is the point overlooked by Mr Max am Ende.

† This point was emphasized by Atcherley and Pearson, *loc. cit.* pp. 5 and 27.

stresses in other cases. Such approximation is hardly likely to be at all close, for the fact that the breadth and height of dams are quantities of the same order, leaves no "small quantity" in terms of which he might expand his functions. Hence *purely* mathematical researches suggest no great hope of real advance in what is notwithstanding an urgent practical problem. It does not seem probable that they would provide any but the roughest approximations to the actual conditions.

(iii) This being the case, experimental investigation arises as the most suitable form of inquiry. Let us admit that Equations (iv) and (v), p. 7, are not justifiable on purely theoretical grounds. We may still inquire to what extent they can give the main portion of the stresses under consideration. If they do, we shall not be troubled that the boundary conditions are not accurately satisfied. We shall admit that this is impossible for the usual type of dam without the introduction of corrective terms. The problem, however, reduces to, not what are these corrective terms, but does the linear distribution of normal stress and the parabolic distribution of shear give 80 or even 60 per cent. of the total value? If the normal stress be mainly linear, then the problem for any shape of dam admits of a graphical solution*. Thus if the linear distribution of horizontal normal stress could be roughly justified, it would seem that a practical solution of the problem sufficient perhaps for most purposes would be forthcoming.

(4) *Experimental Work.* The actual arrangement of suitable experiments is by no means an easy task. For the memoir by Atcherley and Pearson wooden models were used, primarily to test the usual theory that a masonry structure should be sufficiently strong to hold together by friction only, so that no dependence should be placed on its elastic shearing strength. If this theory be correct then the models demonstrated at once that the vertical and not the horizontal sections are the source of weakness. The existence of tension not only in the tail, but at the front between substratum and front† was also indicated by these models. When we wish, however, to consider the elastic properties of dams, a new material has to be sought, and at the suggestion of Sir Benjamin Baker we adopted jelly. It took, however, a considerable time to reach a jelly of a suitable consistency. If the strains were to be measurable, the model must be of considerable size, and further it must give a good surface for ruling. After a number of failures we obtained a cream-white material made of gelatine, glycerine and colouring matter, which did not shrink or rapidly produce vegetable moulds like the water gelatine jellies. This was provided by the *Durable Printers' Roller Company*, and was cast in our own forms. After setting in the mould

* This graphical solution is that attempted by Atcherley and Pearson, *loc. cit.* It involves the consideration of the vertical as well as the horizontal sections of the dam.

† The first one or two vertical laminae were *raised* under excessive water pressure.

and extracting, the face to be ruled required to be made smooth with a hot flat-iron, and was then ruled in squares of 2 cm. side, or in the second series of experiments in rectangles 2 cm. $\times$ 1 cm. The base of the model dams was about 45 cms. and their height about 35 cms. The breadth was 9 to 10 cms. The mass of jelly which represented the substratum was about 45 cms. long by 9 or 10 deep. It might have been desirable to have had a larger substratum, but the material was expensive and the extraction of a much larger amount from the mould not unlikely to be accompanied by risk of damage. The fixing of the substratum to a rigid wooden base presented considerable difficulty, cements and glues of various kinds were tried in vain. The application of the load showed that the jelly dam slid over this cement even during the few minutes requisite for an effective experiment or for a photographic exposure*. Ultimately copper gauze was nailed down to a block of wood, this gauze was made very hot, and the jelly placed upon it. The jelly melting ran into the meshes of the gauze and was thus securely fixed to the wood on cooling. A certain amount of blurring of the line-network on the photographs is probably due to the fact that under the loads applied there was still a small amount of plastic change in the jelly—no longer, however, at the base fixing—during the exposure. The exposures were of considerable duration—about three minutes—in order to obtain a clear picture with the lens at small aperture.

(5) *'Plate' Dam and Solid Dam.* Before the actual nature of the loading is discussed, it is well to consider theoretically the question of side support. In the case of a long dam abutting against rigid ends, there can be no shift v of any section parallel to the length of the dam, consequently if we consider a section some way from the terminals the stresses are not a function of y, the ordinate parallel to the length of the dam, and shears like $\widehat{yx}$ and $\widehat{yz}$, in the cross-section of the dam vanish, although of course the normal stress $\widehat{yy}$ is not zero. These considerations lead at once to the Equations (i) on p. 6. If we wish to obtain exactly the same conditions in a model dam, the jelly must be placed between rigid parallel vertical plates, and slip perfectly easily between them. This introduces special experimental difficulties. The exact parallelism of the plates becomes of vital importance, even if the slip of the jelly between them be assured by well oiled surfaces. If parallelism be secured, then the action of the oil on the network of lines, and the difficulties of photographing or of optically measuring their changes through the glass and oil layer, as well as the greater difficulty of applying the pressure on the front of the dam, become obvious. It seemed therefore worth while inquiring how far the conditions as to strain

* The sliding of the jelly as a whole over its cement fixing was very marked in 'shadow' photographs, which it was thought at first might be useful in measuring the change of strain with increasing loads.

differ, when we take a sheet of elastic material with free side faces. In fact when we consider a vertical plate with base fixed, subjected to no face stresses, and a normal edge stress on a part of its edge only.

Clearly with the axes z vertical x horizontal in the plane of the plate, and y perpendicular to the plane of the plate, we shall have on the two faces:

$$\widehat{yx}=0,\quad \widehat{yz}=0 \quad \text{and} \quad \widehat{yy}=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(\text{viii}).$$

But if these conditions hold at both faces, the values of these stresses can hardly assume sensible values throughout the plate. They are in fact the assumptions usually made in the theory of plates*. But if we adopt zero values for these stresses for the body of the plate, we have:

$$\left.\begin{aligned} \frac{d\widehat{xx}}{dx}+\frac{d\widehat{xz}}{dz}&=0 \\ \frac{d\widehat{xz}}{dx}+\frac{d\widehat{zz}}{dz}+\rho g&=0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(\text{ix}),$$

stress equations absolutely identical with those for the actual dam in (i).

Our stress-strain equations will now be:

$$\left.\begin{aligned} \widehat{xx}&=\lambda(s_x+s_y+s_z)+2\mu s_x \\ 0&=\lambda(s_x+s_y+s_z)+2\mu s_y \\ \widehat{zz}&=\lambda(s_x+s_y+s_z)+2\mu s_z \\ \widehat{xz}&=\mu\sigma_{xz} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(\text{x}).$$

Whence eliminating s_y we find:

$$\left.\begin{aligned} \widehat{xx}&=\frac{2\mu\lambda}{\lambda+2\mu}(s_x+s_z)+2\mu s_x \\ \widehat{zz}&=\frac{2\mu\lambda}{\lambda+2\mu}(s_x+s_z)+2\mu s_z \\ \widehat{xz}&=\mu\sigma_{xz} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(\text{xi}).$$

For the actual dam $\widehat{yy}$ is not zero†, but s_y is zero, and we have accordingly:

$$\left.\begin{aligned} \widehat{xx}&=\lambda(s_x+s_z)+2\mu s_x \\ \widehat{zz}&=\lambda(s_x+s_z)+2\mu s_z \\ \widehat{xz}&=\mu\sigma_{xz} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(\text{xii}).$$

* See Todhunter and Pearson, *History of Elasticity*, vol. II. § 386.

† Messrs Wilson and Gore, *Engineering*, Aug. 4, 1905, in their paper on *Stresses in Dams*, put it zero, hence the vertical and horizontal pressures they calculate have no direct application to real dams.

If therefore in (xi) we put $\lambda' = 2\mu\lambda/(\lambda+2\mu)$, the stresses will be identical in form, except that we have λ' for λ. From (xi) we find:

$$\left.\begin{aligned} \frac{\widehat{xx}+\widehat{zz}}{2(\lambda'+\mu)} &= s_x + s_z \\ \frac{\widehat{xx}-\widehat{zz}}{2\mu} &= s_x - s_z \end{aligned}\right\} \qquad \text{(xiii).}$$

Now if w be the vertical, u the horizontal shift in the plane of the plate:

$$\widehat{xz}/\mu = du/dz + dw/dx,$$

$$\frac{d^2}{dxdz}\left(\frac{\widehat{xz}}{\mu}\right) = \frac{d^2}{dz^2}(s_x) + \frac{d^2}{dx^2}(s_z).$$

Whence:

$$4\frac{d^2}{dxdz}\left(\frac{\widehat{xz}}{\mu}\right) = \left(\frac{d^2}{dz^2}+\frac{d^2}{dx^2}\right)\frac{\widehat{xx}+\widehat{zz}}{\lambda'+\mu} + \left(\frac{d^2}{dz^2}-\frac{d^2}{dx^2}\right)\frac{\widehat{xx}-\widehat{zz}}{\mu} \qquad \text{(xiv).}$$

Now the general solution of (ix) is

$$\left.\begin{aligned} \widehat{zz} &= \frac{d^2V}{dx^2} - g\rho z \\ \widehat{xx} &= \frac{d^2V}{dz^2} \\ \widehat{xz} &= -\frac{d^2V}{dxdz} \end{aligned}\right\} \qquad \text{(xv).}$$

Substituting these values in (xiv) we find μ and λ' disappear and give us:

$$\left(\frac{d^2}{dx^2}+\frac{d^2}{dz^2}\right)^2 V = 0 \qquad \text{(xvi).}$$

Thus the stresses in the plane of the plate and in the actual cross-section of a long dam are found by precisely the same Equations (xv) and (xvi), and since they will have the constants of the solution determined by exactly the same boundary conditions (if we assume that the stress values over the 'base' are among these conditions) it follows that the determination of the stresses in the cross-section of a dam and in the plane of the plate are identical problems, and we may experiment, with limitations to be now noted, on the narrow strip of dam without side supports.

Turning to Equations (xi) and (xii) we see that since the shearing stress is the same, the slide or change in angle, $=\sigma_{xz}$, to be measured from the distorted forms of our network of horizontal and vertical lines, remains the same. In other words the question mooted as to the distribution of shear over the horizontal sections of the dam can be dealt with by aid of experimental 'plate' dams without side supports.

If we want, however, to find the distribution of the normal stresses $\widehat{xx}$ or $\widehat{zz}$

in the plane of the cross-section which are absolutely independent of the elastic constants of the material, we must not measure the stretches s_x and s_z in the model sectional dam. These will depend on different elastic constants in the 'plate' and solid dams. We must deduce $\widehat{xx}$ and $\widehat{zz}$ from Equations (ix), which will then be also the values for the model of a true dam; we should thus be able to discover whether $\widehat{zz}$ is linear in x, or $\widehat{xx}$ can ever be positive, i.e. the dam be subjected to horizontal tension. We can then proceed to find s_x and s_z from (xii) and not from (xi). It is not possible on model jelly dams to measure the values of the stretches (or usually squeezes) s_x and s_z locally with any real accuracy*. We have already pointed out that the actual stresses in no way depend on the elastic constants of the material. Now for a jelly λ is very large as compared with μ, in fact Poisson's ratio $= \frac{1}{2}\lambda/(\lambda+\mu)$ is very nearly $\frac{1}{2}$. Thus we may write (xi)

$$\left.\begin{aligned}\widehat{xx} &= 2\mu\,(2s_x + s_z)\\ \widehat{zz} &= 2\mu\,(s_x + 2s_z)\end{aligned}\right\} \quad \text{..............................(xvii).}$$

If the normal stresses on the horizontal sections are to be *pressures*, $\widehat{zz}$ must be always negative, and therefore either s_x and s_z always negative, i.e. squeezes, or else if s_x be positive it must be less than twice the vertical squeeze. Now undoubtedly points of horizontal stretch can be found in the model sectional dams, but it is not easy to determine whether at such points s_x is greater than $-\frac{1}{2}s_z$, which is the condition for horizontal tension in the dam. The difficulty will be easily recognised, when once it is realised that it is the local values of the

* In a paper in *Engineering*, Aug. 4, 1905, Messrs Wilson and Gore measure s_x and s_z locally on a small india-rubber dam. I am not able to realise how they have surmounted the experimental difficulties, especially with a dam which was locally loaded with a system of isolated loads. Again their equations for finding the pressures from the stresses are incorrect, if applied to a real dam. Further the elastic constants for masonry are very different from those for india-rubber or jelly and the stresses calculated from the stretches will not correspond to the stresses in a masonry dam. We must invert the process and get at the stresses in the model *first*, and these will enable us to find the stresses in the actual dam and from these the strains in the actual dam. Messrs Wilson and Gore distributed the shear over their base linearly. This constraint of the shear to a definite type seems to destroy at once the application of their results to actual dams. It is perfectly true that a solution can be found for a triangular dam, if we assume the shear thus distributed (Atcherley and Pearson, *loc. cit.* p. 25) and in this case (their Case I, of course) the shears on all horizontal planes should have been linear, the pressures on these planes also linear, and the horizontal pressures constant for each height. Their diagrams Figs. 5, 8 and 9 show that these results are probably true within the limits of their method of experimenting. A little consideration, however, will show that the influence of their local loading was of the order of the total effect at each point, and considering the extreme difficulty of accurately measuring stretches in a model only 7″ × 4″, with 27 local loads applied to it, the smoothness of their curves is not convincing. However this may be, the main point to be remembered is that linear distributions of shear are not in the least approximated to, when a dam of any form abuts on to a mass of approximately the same homogeneous character. This oversight of the treatment of triangular dams by M. Lévy and Professor Unwin had been previously referred to by Atcherley and Pearson, *loc. cit.* p. 25.

stretches and squeezes which are required, and accordingly that it is the extensions and compressions on lengths of some 2 cms. at a maximum which have to be measured. Even with a jelly a very delicate extensometer is needed to measure changes in a 2 cm. length, and we did not believe it in the least likely that reliable results could be obtained. We contented ourselves with noticing the existence of stretches in the substratum and tail of model dams when the dams were overloaded, and marking how these became ruptures when the water pressure was still further increased. The determination, however, of the extensions and compressions in the model dams is unnecessary as far as the main problem is concerned. For the hypothesis of the linearity of normal pressures leads at once to the parabolic distribution of shear on the horizontal sections. We can therefore test the legitimacy of the usual middle-third rule by enquiring whether the changes of angle along the horizontal sections are distributed in curves at all approximating to parabolas.

(6) *General Remarks on Experimental Methods.* Our experiments were made *inter alia* on five model dams:

(*a*) A water gelatine model of a Vyrnwy type dam tested to rupture, and especially designed to bring out preliminary difficulties.

(*b*) A control dam of the Vyrnwy type cast in very stiff glycerine gelatine jelly.

(*c*) A dam of the Vyrnwy type in moderately stiff glycerine gelatine jelly.

(*d*) A dam of the Assuan type in moderately stiff glycerine gelatine jelly.

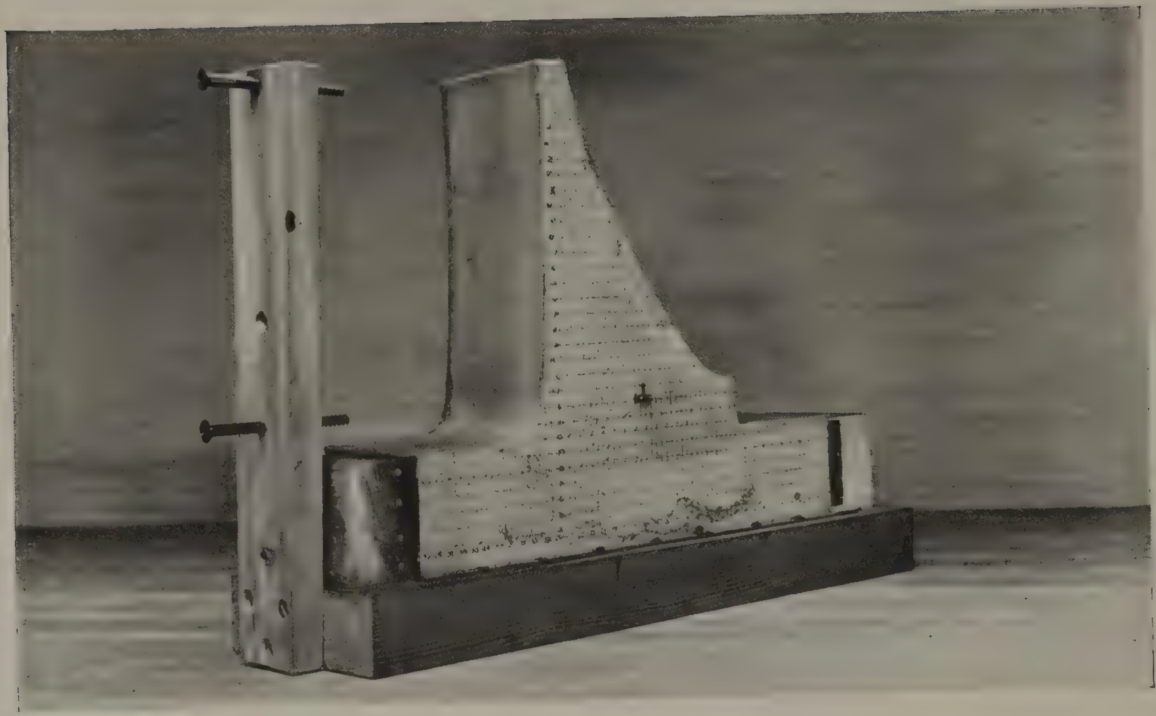
(*e*) A like dam cast a good many months later to be tested solely by optical methods.

The water pressure was applied to the front of the dam in three distinct methods:

(i) By a board pressing against the dam front by means of long strips of small india-rubber piping lying in vertical planes. The board was loaded normally at the centre of pressure of the front of the dam ($\frac{2}{3}$rds down) by a stirup round the dam, and the stirup was pulled by a cord carried over an adjustable pulley and attached to a shot bucket. (See Plate I.) It was assumed that an india-rubber tube compressed between two parallel surfaces by a force not acting at the centre of its length, would remain at all points in contact with the surfaces or its diameter in the plane of loading be changed linearly. In other words the strain of the tube and therefore its pressural resistance would give the linear relation required to mimic a water pressure.

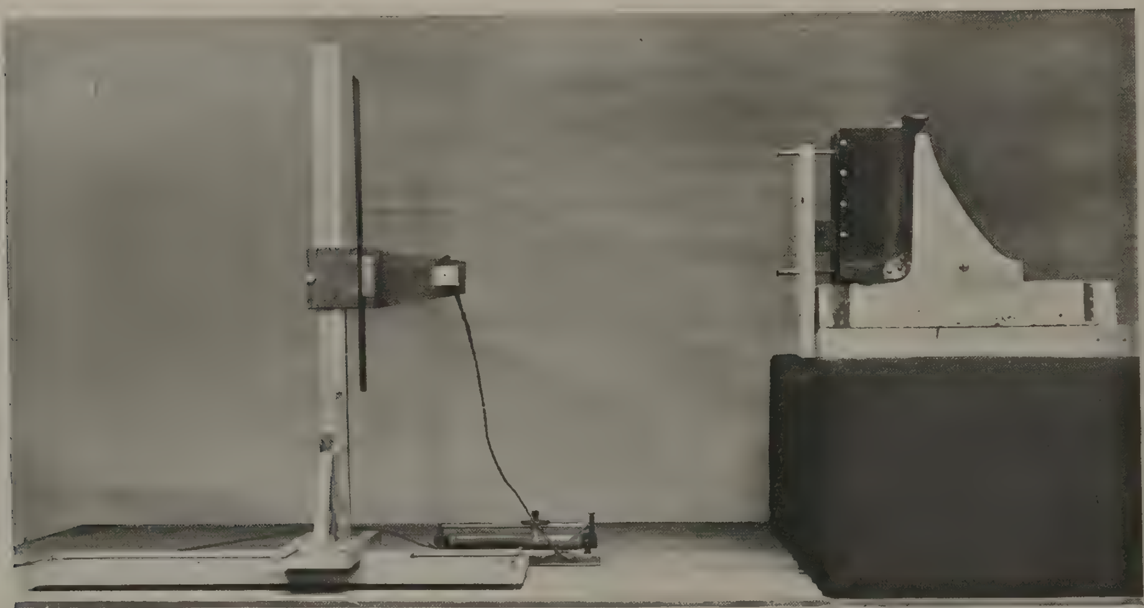
(ii) The board rested directly against the jelly and was pulled at the centre of pressure. It was found that very little difference occurred in the slides along horizontal sections of the jelly whether the board was applied directly or with

Plate II.



Optical Microgoniometer in Position on Dam.

Optical Microgoniometer, directly and by Reflection.



Water Pressure Bag, and Microgoniometer with Scales.

the intervention of the tubing. The frontal layer of jelly itself answered the purpose of the tubes.

(iii) The water pressure was applied by means of an elastic bag resting on the forepart of the substratum and against the front of the dam, the sides of the bag being rigid and supported independently of the dam. See Plate II, Fig. 3.

The advantages of (iii) are that we get an absolutely true distribution of pressure, and we obtain it not only on the front but also on the forepart of the substratum, whereas in (i) and (ii) weights have to be piled upon the forepart of the substratum to roughly represent the water pressure on this part; there is usually also an unloaded strip between the pressure board and these weights; for the board has to be rounded off in (ii) to prevent its cutting into the jelly. On the other hand we have to remember that the jelly has not the specific gravity of masonry, and accordingly as we cannot vary the water pressure easily and satisfactorily, relative to the density of the model dam, we have to go through rather elaborate calculations to adjust the weight of dam to the water pressure. Further we cannot test to rupture, and with the size of dams used the strains must be so slight that they can only be measured by a delicate optical process.

By aid of (i) and (ii) we were able to exaggerate the water pressure* until the dam collapsed, or in the case of very stiff jellies until the stretches were roughly measurable.

(7) *Nature of Collapse of Model Dams.* Another point has now to be considered. If we suppose the front of the dam to meet the top of the sub-

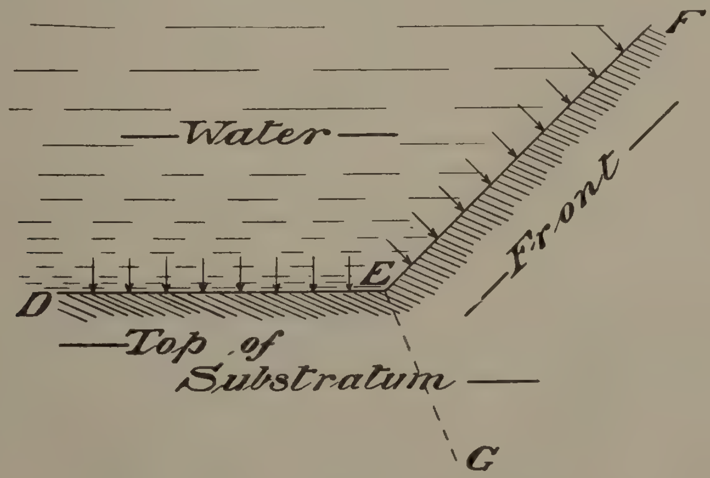


Fig. 1.

* It may even be doubted in any case where a dam is not the side of a *still* reservoir, but is built across a running stream, whether the hydrostatic pressure is the proper value to take for the load on the front. The pressure, when the water is flowing over the top or through the sluices, must be really hydrodynamical, and the effect of rotating masses of fluid against the front may be considerable.

stratum at an angle, for example at the re-entering angle DEF in Fig. 1, then it is easy to show* that any system of normal pressures along DE and EF leads to infinite stresses at a theoretical angle E. Even if the angle E be rounded off, it follows both theoretically and experimentally that the tensions perpendicular to the line EG will be very great. The large stretches across the line EG, even when the angle is rounded off far more than is likely to be the case in actual practice, are one of the marked features of the experimental dams. This problem of rounding off the angle was the one for which Dam (a) was specially used. Even after considerable changes had been made in the moulds, far beyond the rounding in actual dams, the weak point of the dam was invariably found to be at its juncture with the substratum along the line EG. This was true notwithstanding the rounding of the pressure board or the absence of pressure at all along a narrow strip in the neighbourhood of E. The weak feature in the model dams was always the large stretches in the substratum perpendicular to EG. There can be small doubt that for fairly homogeneous material in the case of both dam and substratum, EG gives the plane of weakness. Of course other sources of *local* weakness—possibly more important sources—will actually occur when we deal in practice with mixed masses of stone and concrete, and irregular rock substrata. Nevertheless we say with considerable confidence to the practical constructor, keep your eye on what is happening or may happen along the plane EG.

Fig. 2 shows the actual rupture history of a large jelly dam of the Vyrnwy

Fig. 2.

* Pearson: *History of Elasticity*, Vol. II, Art. 1711.

type—Dam (*a*). The first stage is excessive stretch perpendicular to *EG* in the neighbourhood of *E*; then a slight crack began at one or other vertical face and spread ultimately into a tear right across the breadth of the dam. This tear runs towards *G*. Before however the stretch perpendicular to *EG* at *G* sufficed to tear the jelly at *G*, it tore the jelly from the cement, or tore the cement itself along the rigid base of the substratum. Then arose signs of tension in the tail and in the substratum under the tail. These signs are preparatory to the tail tearing open up a vertical section, and finally there occurs the toppling over backwards of the whole forepart of the dam over the separated part of the tail.

Of course in the case of an actual dam the whole history of the collapse would be expedited by the lifting effect of the water getting into the crack *FG*. In broad outline the above is the nature of the experimental collapse of dams.

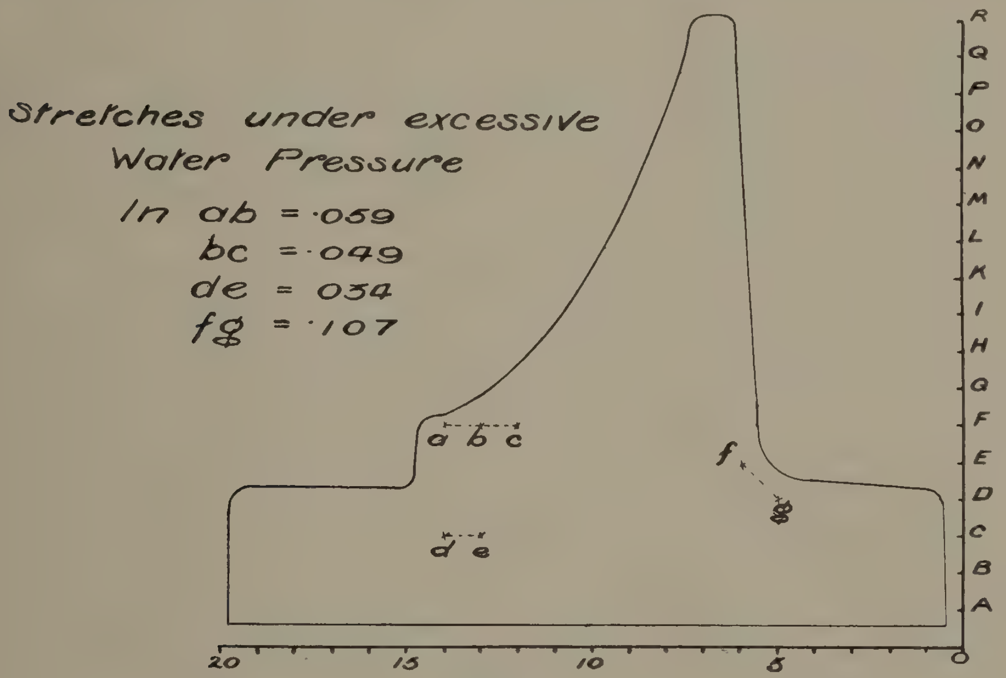


Fig. 3.

The stretch in the tail which arises with exaggerated water pressure is a precedent to collapse, but it is *not* the most important precedent; the initial weakness is in the substratum and not in the dam itself. The actual nature of the collapse, (*a*) the lifting of the front of the dam as it cracks along *EG*, (*b*) the vertical crack of the tail and (*c*) the toppling over of the forepart of the dam over the fractured portion of the tail is precisely the same as occurs in the wooden model built up of vertical laminae described in the first memoir. In other words jelly models show that the dam built up of vertical laminae and not that built up of horizontal laminae more characteristically illustrates

the nature of the final collapse of dams in general. Such wooden models form quite effective lecture illustrations of what takes place when it is not possible for ordinary educational purposes to test large jelly models up to rupture.

With the much stiffer models made of glycerine and gelatine, it was not feasible with the apparatus at our disposal to test up to rupture. To control, however, the results obtained for the Vyrnwy Dam (a), measurements were made of the stretches in the Assuan type Dam (d) under excessive water pressure—as near rupture as we could go. The maximum stretches, see Figure (3), were found to be: in fg perpendicular to EG of as much as ·107; in ab of ·059; in bc of ·049 and in de of ·034. These indicate exactly the same states of over-strain and collapse as occurred in the corresponding parts of the Vyrnwy model.

If actual dams correspond in any way to our models, and there seems no reason why they should not, increased protection against collapse would be provided by (a) very markedly rounding off the angle at E, which might sensibly reduce the available space on the reservoir side, (b) armouring* the dam perpendicular to the plane EG and possibly to a lesser extent the tail. Such a process could hardly fail to strengthen sensibly such dam types as are dealt with in this paper and would probably allow of a reduction in material for a given height of water.

(8) *First Series of Experiments. Slides read directly.* These experiments were made on the Vyrnwy type Dams (b) and (c), and on the Assuan type Dam (d). They were each twofold in character, namely the slides were determined with a water pressure proportional approximately to the weight of the jelly, and again with exaggerated or excessive water pressure. The very stiff Vyrnwy Dam was used to find what difference, if any, arose from altering the elastic constants of the material—clearly there should be on the theory developed, pp. 10—13, no difference at all.

We know that the shearing stress is proportional to the slide†, and that this stress is the same for the 'solid' dam and the 'plate' dam. Hence if we ascertain the distribution of the slide along any horizontal section, we have really on some scale the curve of shear distribution. To determine the scale all that is needful is to know the total shear on the section—which is of course the horizontal component of the total water pressure above the section; this total shear is the *area* of the curve of shear, i.e. the curve of slide, if its ordinates be read in the required scale of shear, which is thus determined. Now we shall begin by frankly admitting that, even in a substance like jelly which distorts so sensibly, the small angles on which the slide depends are not easy to measure

* Pearson: On the Stability of Masonry Dams, *Engineering*, July 14, 1905.

† In an isotropic material the shear = slide modulus × slide, and the slide may be taken as the change of angle or the tangent of this change of angle indifferently.

and that no great exactitude is possible. In this first series of experiments the jelly after its face had been ironed flat* was ruled with a network of lines, perpendicular to the base of the substratum and parallel to it. This was done with the jelly resting on the other parallel face. Now the models were four inches thick, and weighed about 18 lbs. Hence a certain amount of distortion arose merely from the weight of the jelly even when it was placed horizontally. Further it was not possible to insure absolute accuracy in ruling a jelly with vertical and horizontal lines 2 cms. apart. It was done as carefully as we could do it. Next, when the dam is erected into the vertical position after ruling, the network of lines ceases to be rectangular and if kept in the vertical position for some time a certain amount of set takes place. Thus even before the application of the water pressure we have apparent slide (or change of angle from 90°) in our 2×2 cm. squares due to four different sources: (*a*) the want of absolute truth in the ruling, (*b*) the fact that the ruling took place with the weight acting perpendicular to the face and producing a certain distortion, especially near the contour of the face, (*c*) the distortion due to the elastic effect of the weight of the vertical jelly, and (*d*) the set effect due to the weight of the jelly or to some shrinkage when the jelly was kept for any time. Of these four sources of distortion we do not think that (*a*) and (*b*) while probably sensible are relatively large. (*d*) is, however, more important, and it is not easy to determine its magnitude. Suppose the jelly to be photographed in a vertical position without water pressure loading, then the angles of the network if measured will not give simply by their subtraction from $\frac{1}{2}\pi$, the elastic slides due to the weight of the dam, (*d*); and to a much less extent (*a*) and (*b*), affect the result. If the water pressure loading be now applied, the slides for the same reason will not give the pure elastic effect of weight and water pressure. If, however, we photograph on the *same* occasion the dam with and without the water pressure loading, we shall obtain by subtracting the angles the slides due to water pressure alone. If we measure the angles under water pressure and weight we shall, by finding the change from right angles, have an approximate value of the shear distribution, subjected, however, to more or less error due to (*d*) and some irregularities resulting from (*a*) and (*b*).

But the hypothesis in practical use which we have got to test is the linear distribution of pressure over the horizontal sections, and we have seen that this hypothesis inevitably leads to a parabolic distribution of shear. Now the hypothesis is *absolute*, it states no limitation as to the relative densities of masonry and water for which it may be supposed to be true. In fact it is a common custom to construct by means of this very hypothesis the line of resistance of the dam when the reservoir is empty. But if the hypothesis be true for empty

* This is a delicate task, but with caution an excellent surface for ruling can be prepared.

reservoir and for full reservoir, it follows at once that it must hold for water pressure only. In other words, we may investigate the changes in angle produced by changes in the water pressure loading only, and there should follow a parabolic distribution if the hypothesis of linear pressures over the horizontal sections be true. We can of course consider and have actually done so the total angular changes from the original network, but as want of parabolic distribution might then be hastily asserted by some to be due to (a), (b) or (d), we can consider the simpler problem of the defect in linearity of pressure due to water pressure loading only. In each case three photographs were taken: (i) of the vertical dam without water pressure, (ii) of the strained dam under moderate water pressure, (iii) of the overstrained dam under excessive water pressure. In cases (ii) and (iii) the distribution of the shear over the horizontal sections was found to be of the same general character.

In the case of the Vyrnwy type dam it was cast, as we have seen, several times, partly to test various jellies, partly to test the right amount of rounding at "E," partly to select a suitable degree of stiffness; and partly to test how far the stiffness could possibly modify the results. The distribution of shear on the dam base in the early casting was in reasonable agreement with its value as obtained from the more complete measurements on later castings. Further, a photograph of this dam was taken in a horizontal position soon after casting and ruling. The defect from accurate ruling noted in (a) was thus measured; it was found to be of quite a different order of magnitude to the angular changes referred to as (c) and (d).

(9) It will be seen that the problem in this first series of experiments reduces to the measurement on a photograph of the angle between two short lines originally 2 cms. in length*. This is not, however, an easy task in itself. At first the photographs were looked at through a filar position micrometer belonging to an equatorial telescope, but the measurements having to be taken in a room where other persons were at work and with traffic round the building, the absolute steadiness requisite could not be obtained. Then drawing board methods were adopted, and a fairly reasonable accordance found when two observers determined the same angle. Still it was not easy to find by any drawing board method very accurately the angle between two 2 cm. lines photographed from a ruling on jelly. Accordingly the following instrument, called for convenience a direct microgoniometer, was devised and constructed by A. F. C. Pollard. It consisted essentially of two separate but rigid pieces $\alpha\alpha\alpha$, $\beta\beta\beta$, both formed of sheet celluloid. (See Fig. 4.) The main piece $\alpha\alpha\alpha$ is formed of three pieces cemented together with amyl acetate. The arc BC, part of α, has a

* Whole plate photographs were taken of the dams, which were then enlarged up to sensibly the same size as the models.

scale engraved upon it, and is rigidly connected with the piece DP by the arm E. The piece DP ends in the point P which is the centre of the scale on BC, the radius being 10″. PD and BC lie in the same plane, and the arm E lies in a plane above them. The whole piece $\alpha\alpha\alpha$ can be rotated about the point P

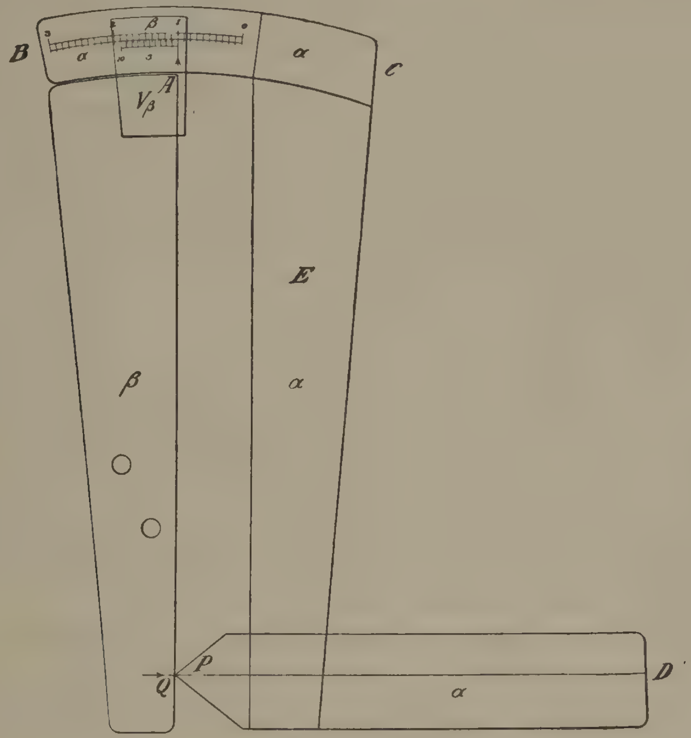


Fig. 4.

when lying upon the photograph with slight frictional resistance. The second piece $\beta\beta\beta$ is simply a radial arm with a straight edge AQ, a definite point Q of which plays against the point P, when the instrument is in use. The radial arm $\beta\beta\beta$ carries a Vernier V. This vernier is engraved on a piece cemented to $\beta\beta$ and so lying in the same plane as E. The main portion of $\beta\beta$ lies in the same plane as BC and DP. Thus the vernier slides in close contact with the scale on BC, being of the same radius and with centre Q. It is engraved on the underside of the piece V, while the scale BC is on the upper face of BC. The scale BC is 3 inches of arc, and is graduated from 0 to 3. When the zero of the vernier is at 1 on BC, then the edge QA is truly at right angles to a line PD drawn on the underside of the piece DP through the point P. The range of the scale BC admits of our reading to an inch on either side of the position 1 on the scale, or to an angle of which the circular measure $= 1/10$ or an angle of $5° \ 44'$ nearly. But the tangent of an angle of $5° \ 44'$ differs from its circular measure by less than ·00034, and as the instrument is only designed to read to ·001 of a radian or ·01″ on the scale, it is clear that it may be used to give the

slides or tangents by angular reading. In no cases, even with the overstrained dams, did the change in the angles of the network exceed the limits of the instrument.

The method of using the instrument will be obvious. The point Q is placed at the point of the network in the photograph at which the slide is to be investigated, with the edge QA in contact with one of the 2 cm. lines meeting at that point. The line PD is then turned round P until it coincides with the second 2 cm. line through Q. The vernier is then read, and its reading less unity and divided by 10 is the change in angle in circular measure, or the slide at the given point. Multiplied by the slide modulus of the material it is the shear the distribution of which has to be found. Of course the line PD may first be fixed and the edge QA turned round P till it coincides with the second 2 cm. line if this be found more convenient.

By aid of this very simple instrument it was found quite possible to rapidly take a number of measurements of any angle, and to some extent the fatigue which so often accompanies the direct measurement of a large series of small quantities was avoided. Photography and the use of the direct microgoniometer* sufficed to demonstrate for both Assuan and Vyrnwy dam types that both along the base of the section and up till there were not more points determinable than fix a parabola, the slide curves were *not* parabolic.

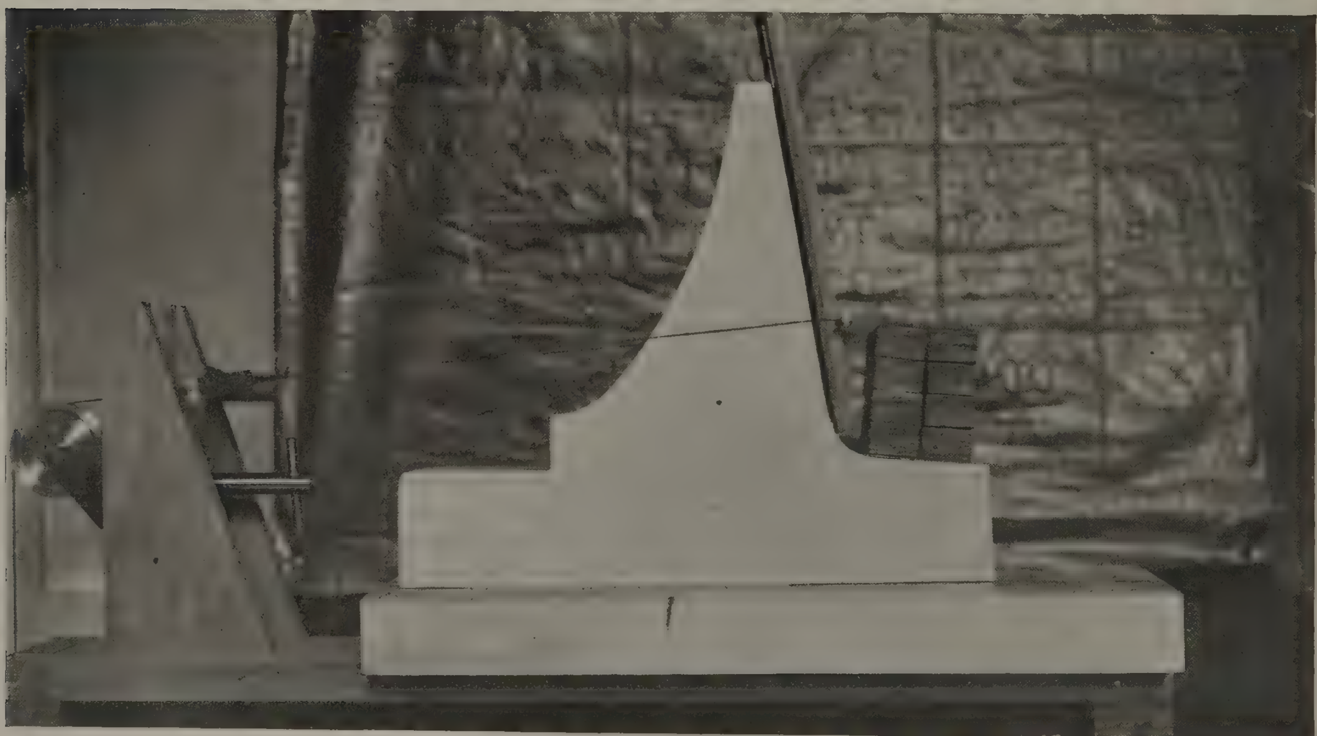
We do not propose to give the whole series of measurements determined by this process, because a more accurate one was used in the second set of experiments. It will suffice to indicate the general results. Plate III, Fig. (i), shows an Assuan type dam subjected only to the slide produced by its own weight; Fig. (ii) shows a second Assuan type dam subjected to excessive 'water' pressure. Plate IV, Fig. (i), gives a very stiff Vyrnwy type dam with the network distorted by the weight of the dam only, Fig. (ii) shows the distortion of a standard jelly Vyrnwy due to moderate water pressure. Many such photographs were taken, including 'shadow' photographs, after intervals of time under the same load to mark the plastic action of the load on the jelly, and immediately under different loads. The photographs on which the angular measurements were taken were some $22'' \times 27''$, being the full size of the models.

Fig. 5 is a key figure to the networks on the Assuan and Vyrnwy models. Curves of slide were plotted for all the horizontal sections of both these dams for (a) weight only, (b) weight and moderate water pressure and (c) weight and excessive water pressure, (d) moderate water pressure only, and (e) excessive water pressure only.

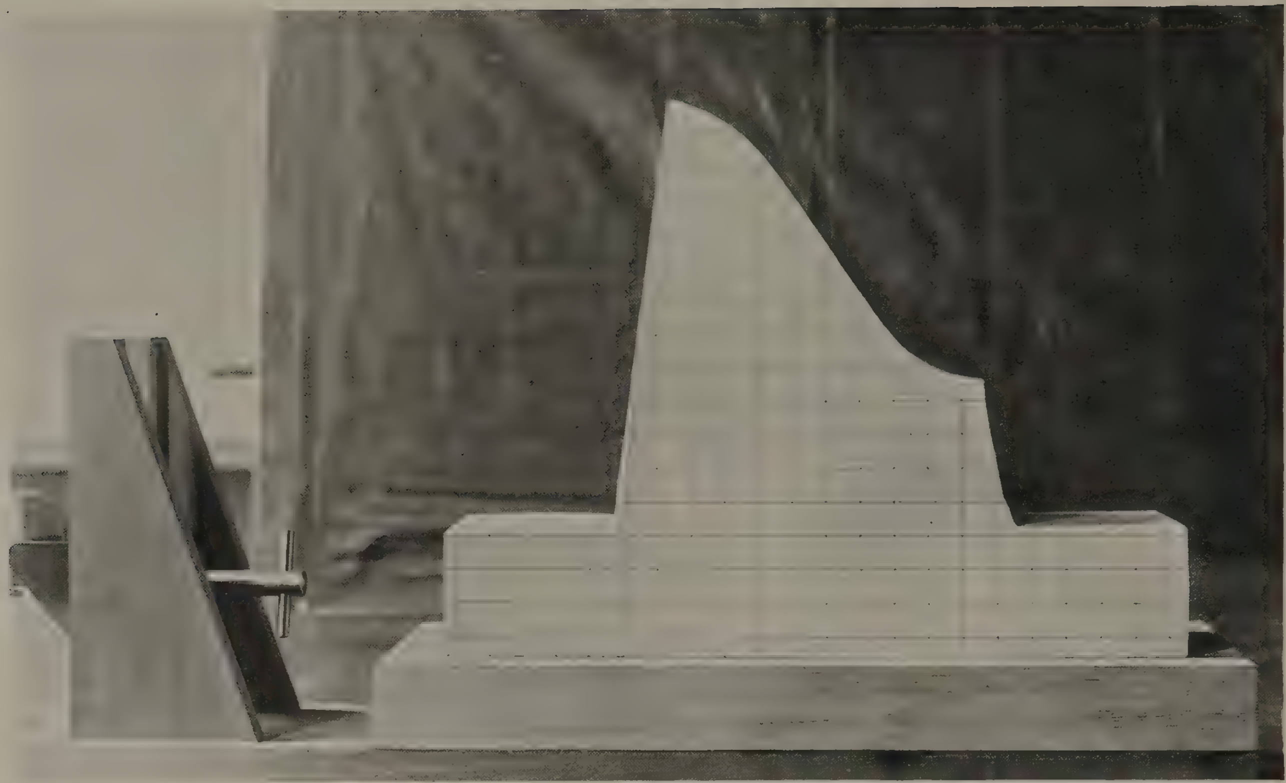
* A considerable number of measurements were taken of each angle, and these were averaged, unless the discordance between values showed a special source of error in one or other, when the angle was carefully remeasured.

Plate III.

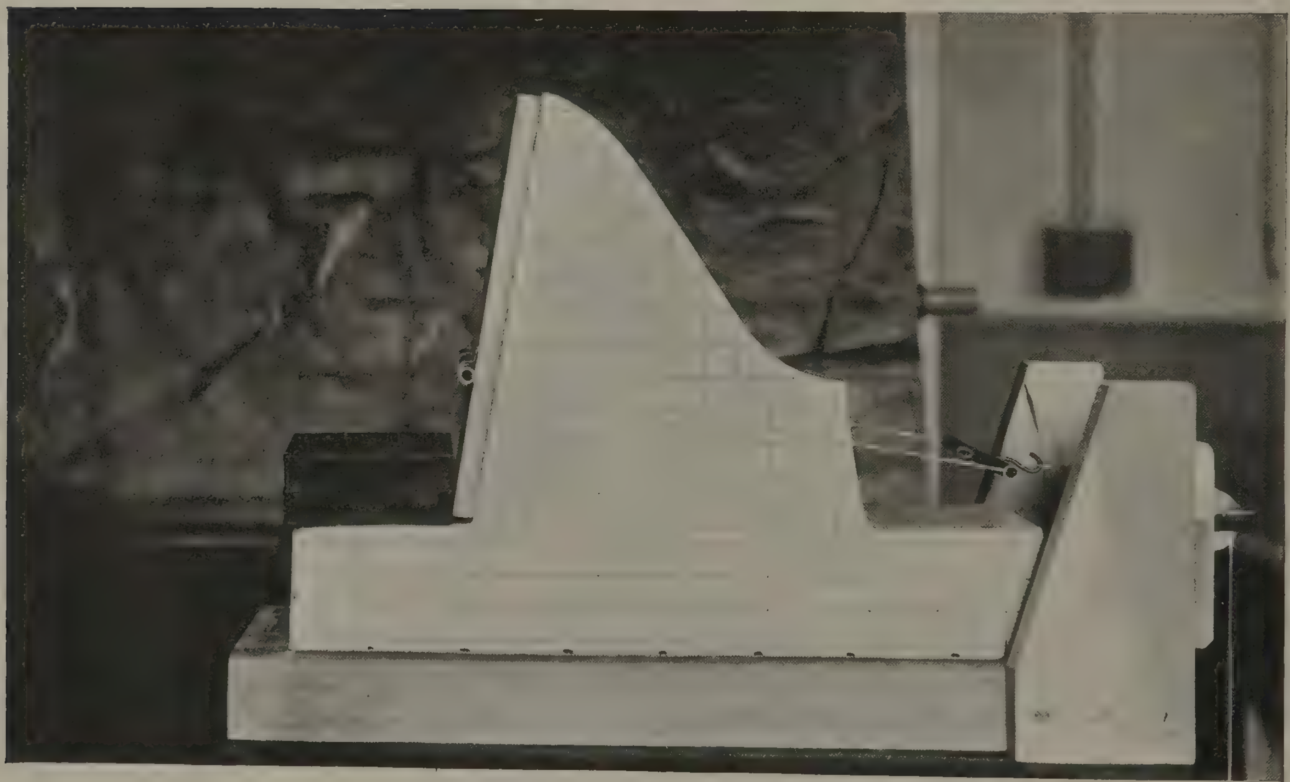
Assuan-Type Dam, Moderate Water Pressure.



Assuan-Type Dam, Exaggerated Water Pressure.



Vyrnwy-Type Dam, without Water Pressure.



Vyrnwy-Type Dam, Moderate Water Pressure.

(*d*) and (*e*) were much alike and like what were obtained from the special very stiff jelly Dam (*b*). In the case of (*b*) and (*c*) the actual forms of the curves

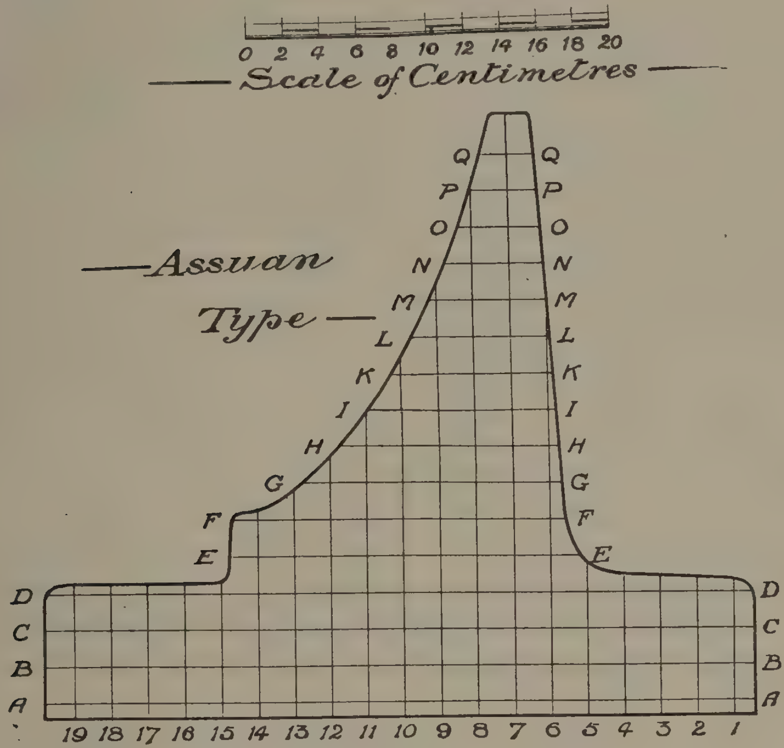


Fig. 5 (*a*).

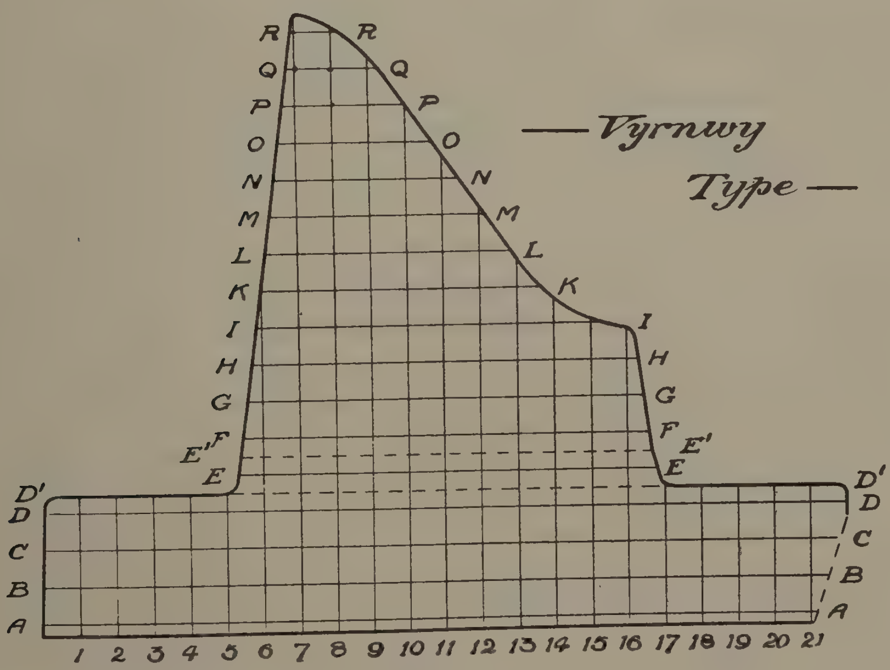


Fig. 5 (*b*).

depend on the relative proportions due to weight and to water pressure, which of course would not be true, if both sources of slide produced a parabola. In fact, with the exception of a few sections near the top of the dams, with too few points to be any criterion, not one of the sections treated in all these ways gives any approach to a parabolic distribution of slide. Making all due allowance for irregularities of ruling, of measurement, and of set, we must still refuse to believe in any parabolic distribution of shear. Plate V gives the slide curves for the section about the base of the Assuan type dam. Fig. 5 above gives the key to the exact position of these sections. It will be seen that the curves while often of an extremely high order are fairly smooth and continuous. Some of course of the irregularities may be due to shrinkages, but certain of them are undoubtedly due to sharp changes in the contour of the dam itself. The parts of the curves dotted correspond to doubtful measurements near the boundary of the face. If we take the curves F, E and D from the verticals 6 to 14, we see that notwithstanding irregularity of measurement, they certainly possess at least the points of inflexion, and it would accordingly be hopeless to represent them even approximately by anything short of a quartic. With this statement all hope of a linear distribution of pressure disappears. If such a curve as that of E in Plate V for water pressure only be compared with that for the similar section G in a total different casting of a dam of the same type treated by a different process of measurement, Plate IX, we see that there is very good concordance between the results. In a general way also the weight curves E, F of Plate V show negative shear towards the front and positive towards the flank, thus coinciding with the difference between the dotted and full curve in G of Plate IX. The effect of weight determined, however, by this first series of experiments is not nearly as good as in the second series, wherein the deflections 'without weight' were obtained by floating the dam just before measuring its weight deflections, whereas in this earlier method a certain amount of set undoubtedly came in.

Plate VI gives corresponding curves for the Vyrnwy type dam, and a few for a second Vyrnwy type dam of very stiff jelly. In this latter case the pipe backing referred to on p. 14 was used throughout. The horizontal section D of this stiff dam corresponds to D' of the standard Vyrnwy. We see at once from the Plate that the general forms of the curves of the D, E, F sections for the standard and the control dams, although they were made of different jellies, had their water pressures applied in different manners, and were differently fixed to their wooden bases (the standard by wire gauze and the control by cement), are essentially the same. In other words the chief undulations of the slide curves are not due to irregularities of mixing material, or of ruling or of measurement; nor again to deviations due to the manner of fixing the base; they are peculiar to the actual contour of the dam front and flank.

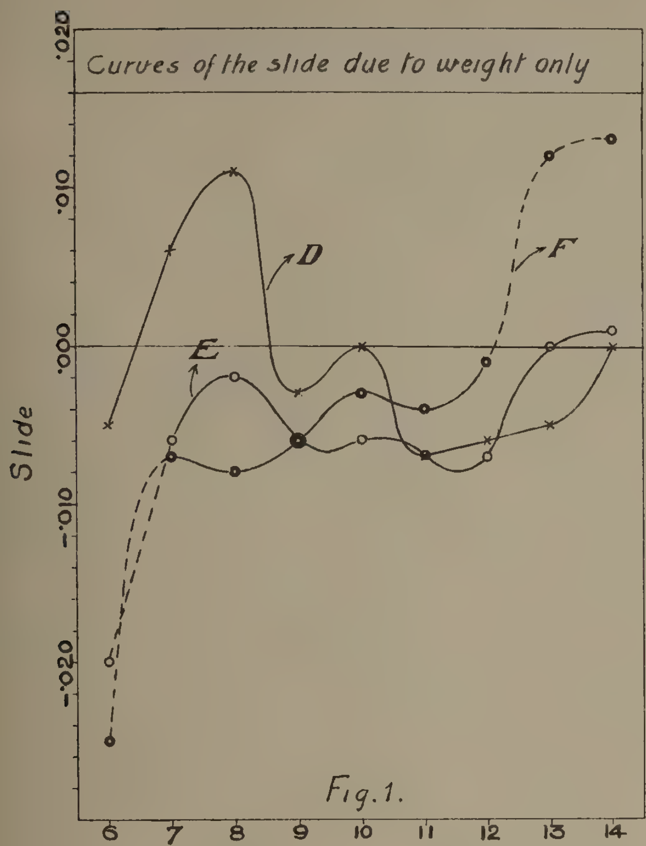


Fig. 1.

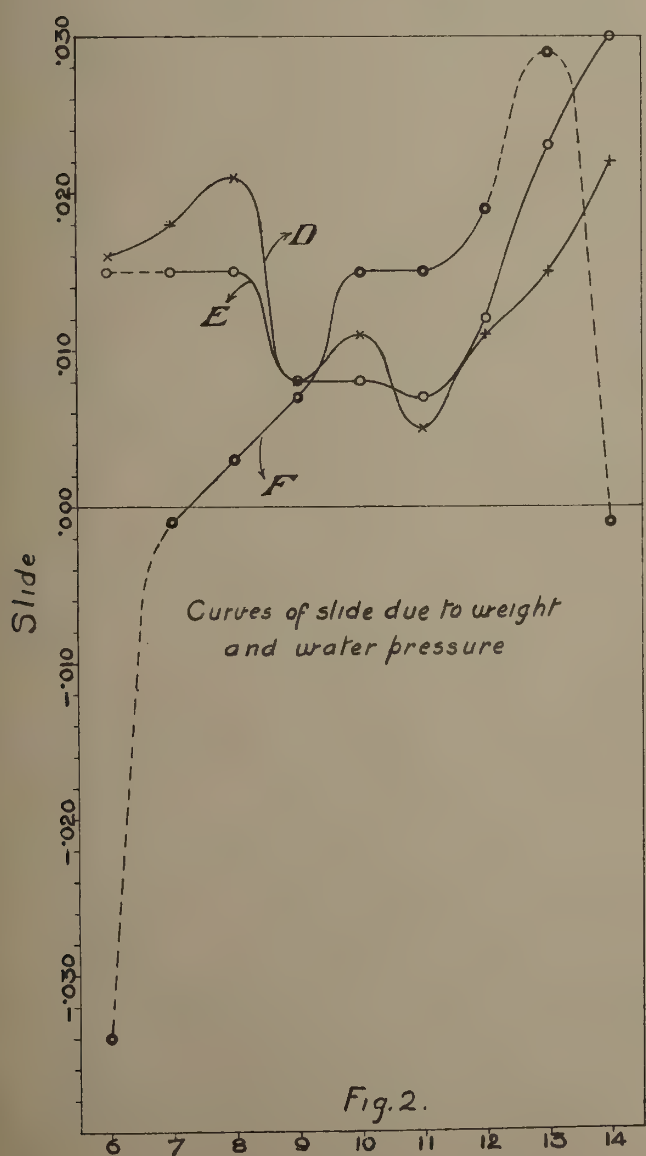


Fig. 2.

Curves of slide on the horizontal sections D, E, F.

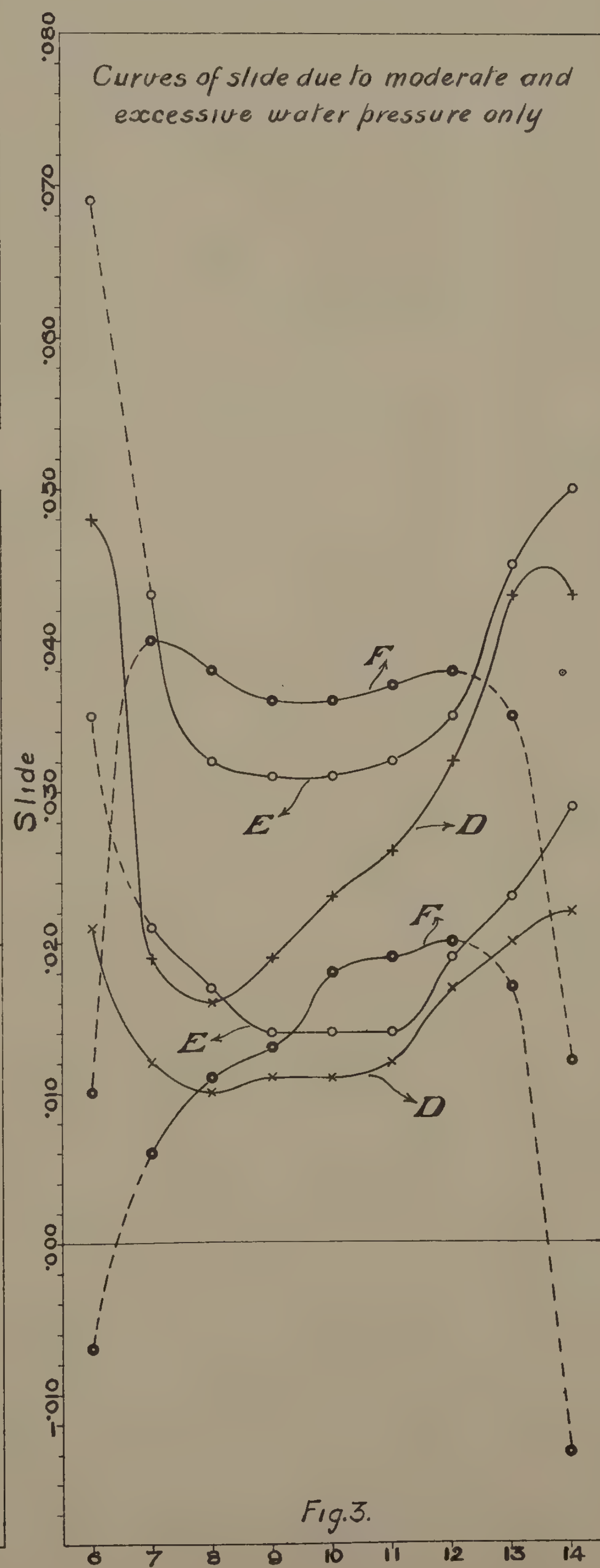
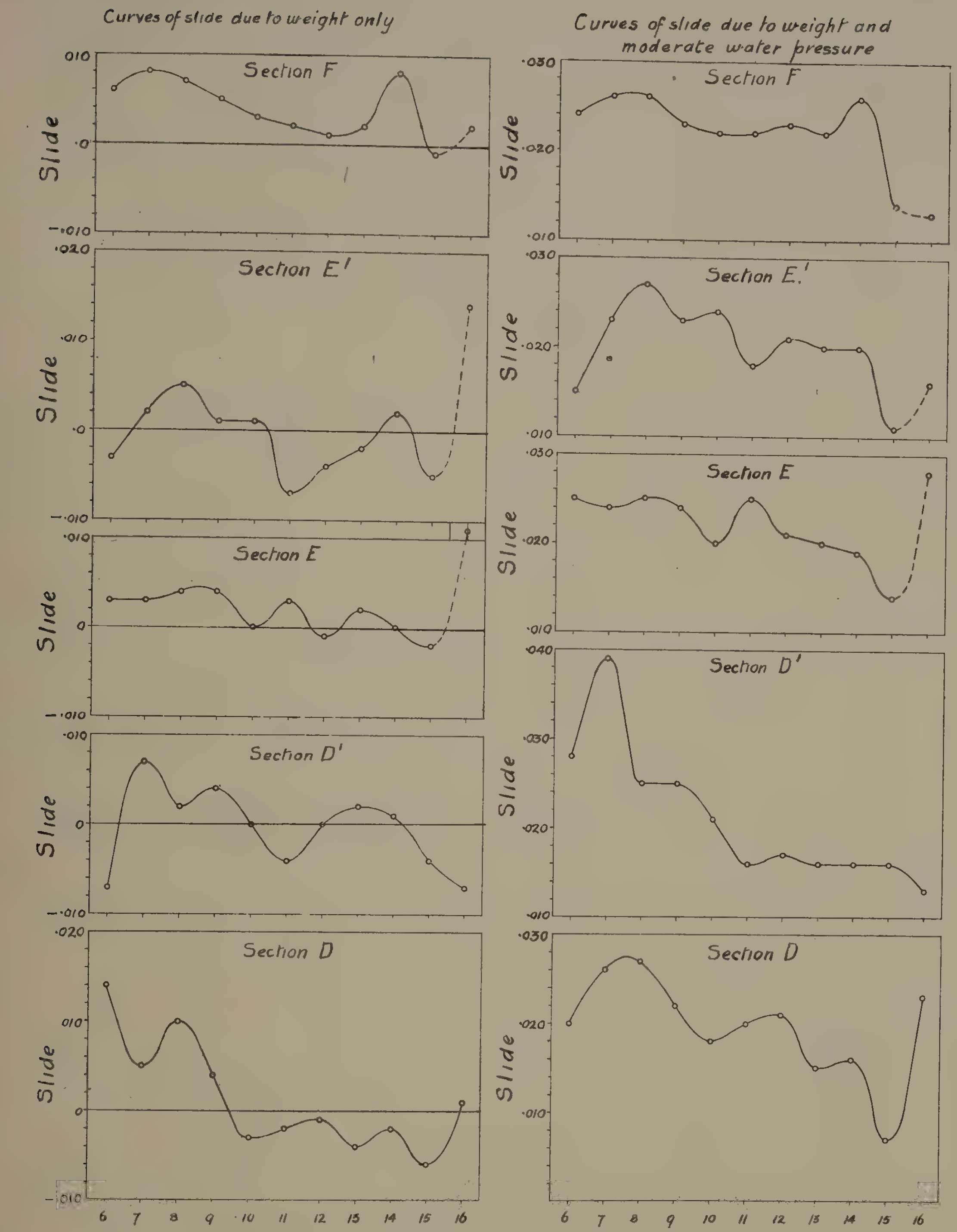


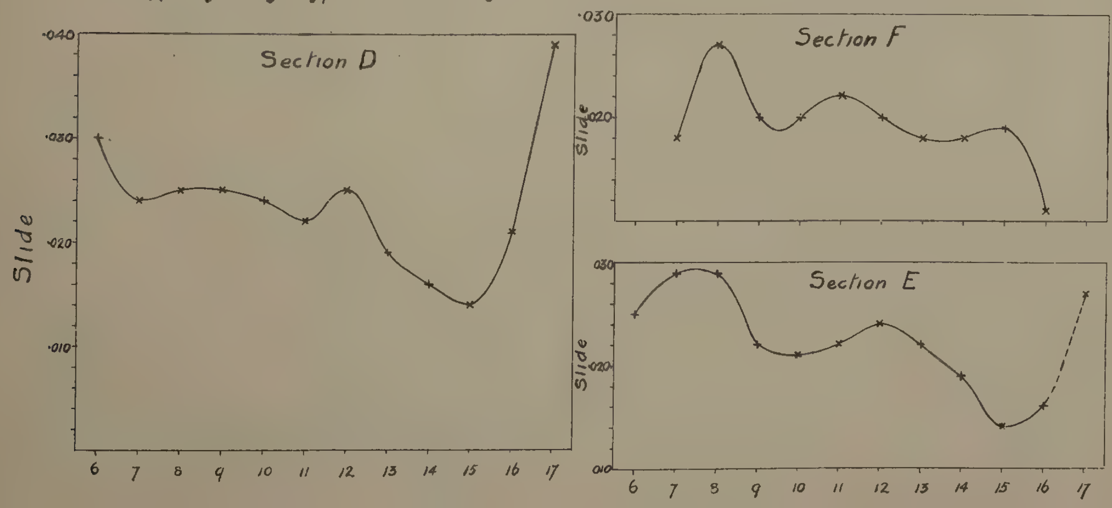
Fig. 3.

VYRNWY TYPE DAM.

PLATE VI.



Stiff Vyrnwy Type Dam weight and moderate water pressure.



While the lowest horizontal section A of the control dam differs somewhat from the A of the standard dam because the dam was not only not fixed to its base in the same manner, but A was also at a different distance from the base, the sections B and C show the same characteristic main undulations as those of the standard. And this correspondence holds right up through the horizontal sections. It has not been thought needful to publish more than the D, E, F curves, but these may be taken as a random sample of the fair degree of concordance in the general shape of the shear curves with different stiffnesses of jelly, and with different methods of loading and of fixing the substratum to the base.

When, however, the dam contours are different, an opposite statement must be made. If we compare the curves for the Assuan and Vyrnwy dams, we see indeed certain points of resemblance, but the comparison will confirm the view that the local contour towards the base of the dam is immensely significant for the nature of the distribution of shear over the basal sections.

Our final conclusion on this first series of experiments must be that: Allowing for every irregularity of measurement and material—and we do not in the least wish to screen any difficulties which arise from these—there is no approach to a parabolic distribution of slide and therefore of shear up at least 2/3 and probably up the whole height of dams of the current forms. In other words, linearity of pressure distribution over the horizontal sections of a dam is a wholly unjustifiable hypothesis. But with the collapse of this hypothesis there falls to the ground also the rule that safety follows if the line of resistance lies in the middle third. This rule provides no safe test for the stability of masonry dams. That most existing dams are stable is a result which we must attribute in the first place much more to experience in the choice of existing contours than to any validity in the customary theoretical test, and secondly, to the use of extremely large factors of safety. The customary theoretical test adds an appearance of safety which is not in any way warranted by our experimental investigation.

The great complexity of the shear distribution on the base, the fact that it varied so from contour to contour led us to deal with the subject *de novo* by a different method, only to reach results in broad lines confirming those already obtained.

(10) *Second Series of Experiments. Optical Measurement of the Slide.*—We have already noted the irregularities and difficulties of the first series of experiments. A further series was, at the suggestion of Mr Pollard, carried out by means of an optical method devised by him. The essential feature of this method consisted in placing three fine needles in the ruled face of the jelly, so that the lines joining two of them to the third form the sides of a right angle. Two light mirrors are carried by this system of needles, so as to be

parallel or nearly parallel when the jelly is unstrained, a beam of light is reflected on to a scale from these mirrors, and the difference in separation of the two parts before and after strain measures the change in the right angle due to a given strain.

The details of the optical microgoniometer devised by Mr Pollard are given in the accompanying figure, Fig. 6. The first figure of Plate II shows the ruled surface of the dam of Assuan type with the microgoniometer in position

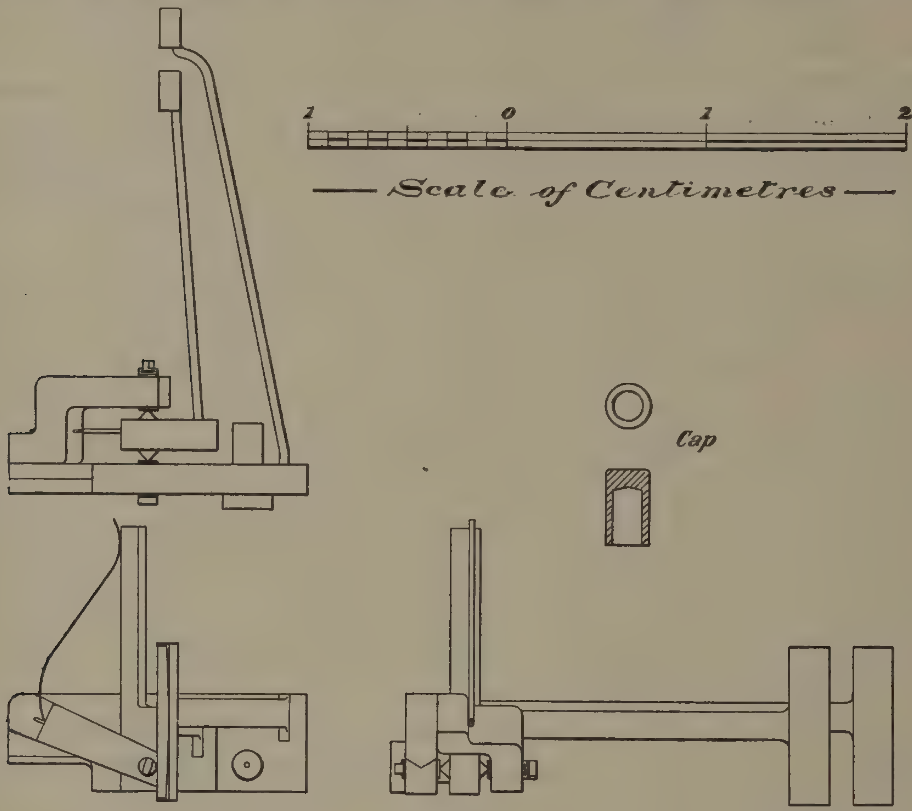


Fig. 6.

on its three needles, the water bag giving the pressure is not yet attached to its support. The second shows on the right an enlarged photograph of the instrument in position on a system of centimetre rulings; on the left is an image in a plane mirror, somewhat distorted. It will be seen that the whole apparatus, excluding the mirror-bearing arms, is not much more than 1 cm. high by 1·5 cm. long. The mirrors are kept in position against the needles by aid of a very feeble spring. The third figure represents the dam with its pressure bag on, and the beam-producing electric light and the reading scale in position*.

The water bag had rigid sides and an elastic front and base which rested respectively against the front and substratum of the dam, being shaped to fit

* In reading the scale a correction had to be made for the obliquity of the 'horizontal' mirror ray. Let BC be the scale, OA the true horizontal, the mirrors being at O and the two

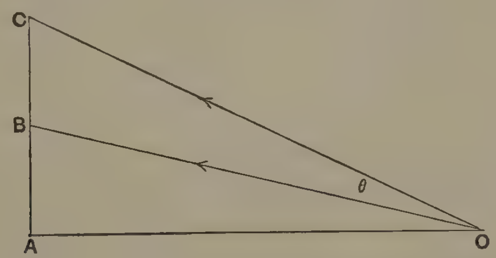
it closely. The sides were supported by a vertical upright independent of the dam. The ends of the substratum were in this case blocked up against rigid supports, while in the previous series they had been left free*. Actually neither arrangement would hold, the substratum would be indefinitely continuous elastic material, but it would probably rise somewhat up-stream and fall away rapidly down stream. It would be difficult to say which arrangement is the more nearly true for any actual dam.

As the water pressure and the weight of the jelly dam had not anything like their relative values in an actual structure, it became of considerable importance to test the effect of the weight only in producing strain. We have seen that this dam was made as the others of glycerine and gelatine. It was accordingly floated with its ruled face horizontal in a bath of glycerine of suitable density†, and the microgoniometer readings for all the angles taken in this position. It will be clear that in this manner a very close approximation was obtained to the unstrained angles‡. The results showed at once that the ruling was, as might be expected, not perfect. The dam was now placed vertically and all the readings again taken. These observations were made on two successive days—the large number taken rendered it impossible to do them all in one—to minimise the effect of set in the jelly. They show conspicuously how large an amount of slide is actually due to the weight of the structure itself. Three days

rays OB and OC. Then $BC = r$ is the reading. Let $AB = k$. It is easy to show that if $AO = 100$ cms., that $\tan\theta$ is given by:

$$\tan\theta = \frac{r}{100} - \frac{1}{100}\,\frac{rk\,(r+k)}{10{,}000 + k\,(r+k)} \qquad \text{(xviii)}.$$

Accordingly $\dfrac{rk\,(r+k)}{100\,\{10{,}000 + k\,(r+k)\}}$ is the correction to be made from the apparent reading to obtain θ. Tables were formed and systems of curves plotted from them to allow of easy determination



of this correction for the values of r and k occurring in the experiments. k was ascertained on each re-fixing of the microgoniometer. The weight of the instrument was about 1·655 grms.

* Of course this difference of fixing modifies the stress curves in the substratum, but does not appear to have much effect on those a little above the base level.

† The specific gravity of the jelly was 1·256, and of the glycerine 1·260, and the dam just floated, although there were some moments of anxiety as to what might happen if it just did not, and the glycerine covered the reticulated face and needles.

‡ The fine needles were inserted once for all, and the weight of 10 such 'diamond' needles = 0·195 grm. They were 3 cm. long and projected 0·6 cm. above the surface of the jelly dam.

were next given up to testing the slides due to water pressure only, the angles being measured before and after the application of water pressure.

We may suppose therefore at this stage two things known—the distribution of slide along every horizontal (and therefore every vertical) section for (i) the weight of the structure and for (ii) the water pressure.

Taking the slides due to weight of the structure if these be plotted along any horizontal section, the ordinates of the resultant curve will on a certain unknown scale represent the shearing stresses along that section due to weight of structure only. The areas of these shear curves represent the total shear on any horizontal section, and this must for all sections above the substratum be zero. Now our specific gravity is only 1·256, instead of the 2·250 of masonry, and the shearing stress must contain this specific gravity as an absolute factor, as it depends on nothing else but the weight. Accordingly we multiply all the ordinates of the curves of shear due to weight only by $2{\cdot}250/1{\cdot}256 = 1{\cdot}79$ before we compound them with the curves due to water pressure. Turning to the slide curves due to weight only we found that when plotted the area was not absolutely zero; this arose partly from errors of observation and partly from the difficulty of determining the finish and start of these curves at the terminals of the horizontal section. At these points, unless the front or flank be vertical, the slide is not zero. We had to content ourselves with the general drift of the curve from the last two observations and a slight raising or depression of the base line to make the area of the curve zero. The accompanying figure gives the index to the ruling, Fig. 7. Adding now the slides due to water pressure only to those due to modified weight only, we have a final curve of shear, on some unknown scale, which would hold for a model dam of the size of the experimental dam made, not of jelly but of stone, under the given water pressure. These curves of shear fully confirm the general result of the previous series of experiments, there is no approach to parabolic distribution of shear over the base, and there is thus no approach to a linear distribution of pressures over the horizontal sections. The rule of the middle third is therefore idle.

The next point which arises is how can these curves, deduced from the model of Assuan type, be applied to an actual structure of very different absolute size? The assumption we make is that the shape of the curves will remain the same, but their scale will be altered, and possibly altered from horizontal section to horizontal section in the real structure*. Accordingly the next stage was to assume these curves to describe the shear over corresponding sections on a full sized dam of the same shape. The areas of these curves must then be the total shears on the corresponding sections of the actual dam. Plate VII represents the actual work on a dam of Assuan type 100 ft. high by 76 ft. deep

* Thus in current theory the line representing the pressure distribution, and the parabola representing the shear, are supposed to hold, whatever be the size or even the shape of the dam.

at the base, the batter being one in sixteen, the weight of a cubic foot of the masonry being taken as 140·4 lbs., i.e. sp. gr. = 2·25. The scale of the drawing is 1 cm. = 4 ft. and 1 cm. = 20,000 lbs. for force, the strip of the dam dealt with being 1 ft. broad. The lines of resistance* are drawn on the plate to show that the dam would according to the usual theory be considered stable.

We now suppose the bases of our curves to represent the number of feet in the actual dam, and further the area of these shear curves to be read off in square feet. Then let the area be multiplied by an unknown quantity K representing lbs. per square foot, then KA has to be equated to the total shear (due of course to water pressure) on the corresponding horizontal section of the dam. The values of the shear were found from the drawing reduced on Plate VII. The values of K found in lbs. per square foot run from 220 at section G down to 140 at section Z,

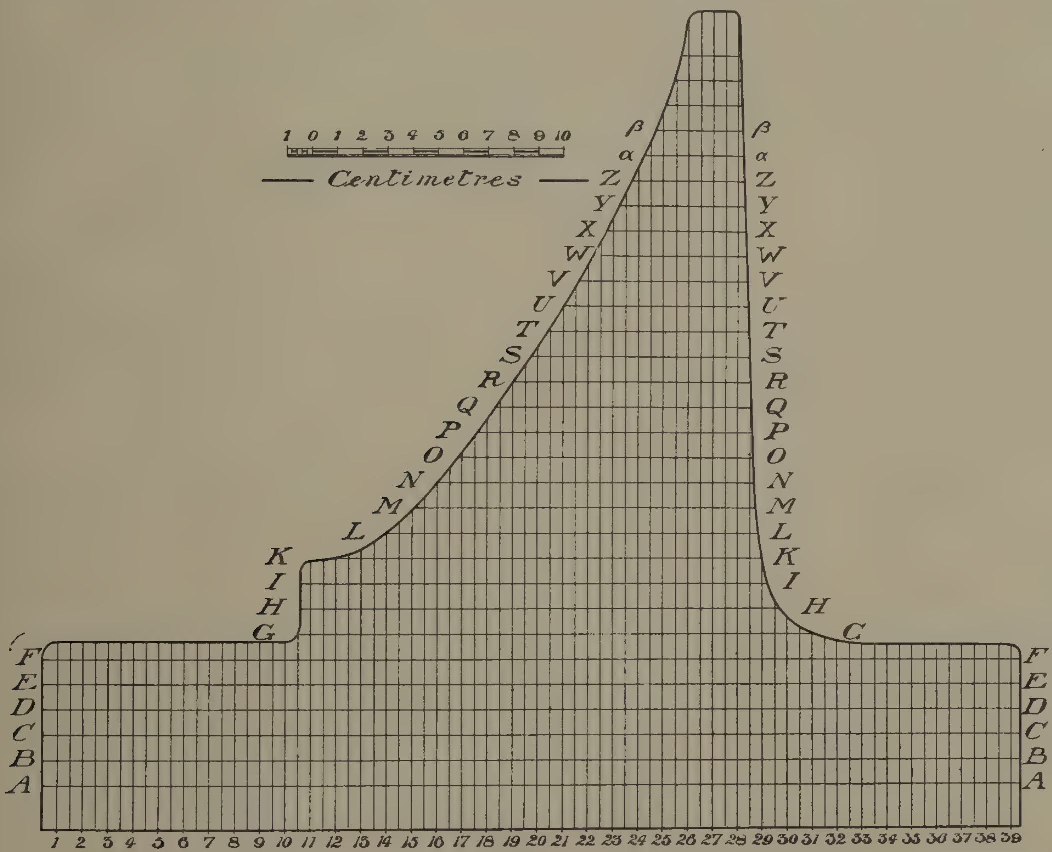


Fig. 7.

and are shown in the diagram, Fig. 8. In other words the actual dam relatively to the smaller model dam increases the relative intensity of its shears in a ratio roughly linear in the depth.

The ordinates were now all recalculated, and the curves on Plates VIII and

* Reservoir empty and reservoir full.

IX determined. These give, in the scale of 1 cm. = 1000 lbs. per square foot, the shearing stresses due (i) to weight only, (ii) to both water pressure and weight over each horizontal section of the dam of Assuan type. It will be obvious from these drawings that neither a parabolic nor a linear distribution of shear represents the facts of the case. In the substratum (Plate VIII) we have distributions of shear due to both weight and water pressure of the form , there being negative shear towards the front and positive towards the flank of the dam. No attempt whatever has been made to smooth these curves, in order that the reader may realise exactly the amount of irregularity due to the methods of experimenting, to the want of homogeneity in the material, and to the complicated shape of the contour of the section. For these sections K was extrapolated* from Diagram Fig. 8, but while the actual value of K modifies the size of these curves,

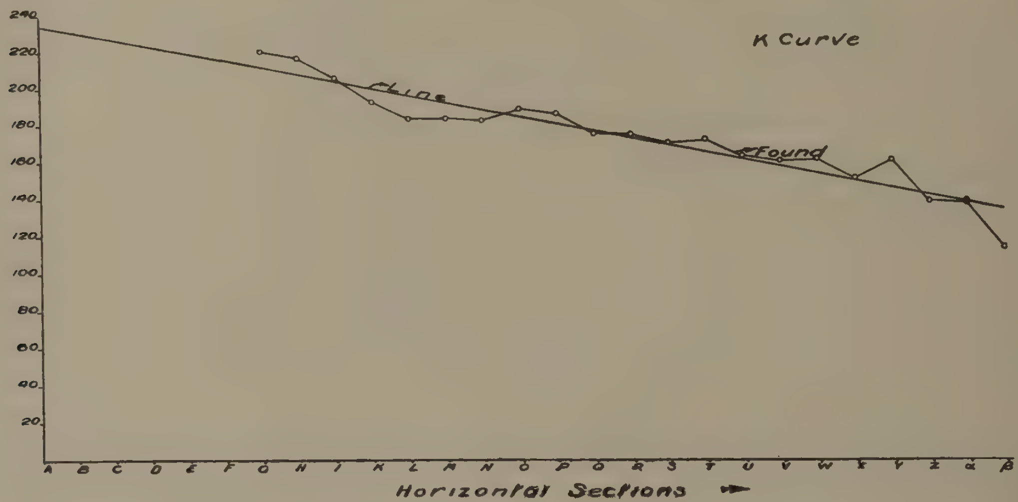


Fig. 8.

it does not of course affect in any way their shape. The 'snake shaped' distribution of shear is maintained up to the horizontal section E. It probably also holds for F, but it was impossible to attach the microgoniometer to verticals beyond 11 and 30 in this case (see Diagram Fig. 7) owing to lack of vertical height for attachment. Accordingly along F (Fig. (i) Plate VIII) the distribution of shear is almost linear. Turning to Plate IX we see that the negative shear to the front of the dam persists in sections G to I. After I the shear becomes sensibly positive. One of the noteworthy results of these curves is the demonstration that the distribution of shears due (a) to weight and (b) to water pressure only (broken line) are quite different in form. We have seen that if the linearity of pressures over the horizontal sections be accepted as a general rule then the shears due to

* The best fitting straight line to the K's for sections G to V was used for extrapolation. Its equation referred to section O as origin was $K = 183{\cdot}513 + 3{\cdot}902x$; x being the horizontal coordinate of Fig. 8, or the vertical height of horizontal section in Fig. 7.

weight and again to water pressure would both distribute themselves according to the *same* form of curve, i.e. a parabola. It will be noted that the character of the shear curves is really wholly different in the two cases. In other words, if there were any meaning in the line of resistance for reservoir empty, it would cease to be valid for reservoir full. As a matter of fact these shear curves demonstrate that neither line of resistance as far as the arguments drawn from the 'middle-third' and linear distribution of pressures are concerned serves any useful purpose at all.

There is most probably a slight negative shear over the very front of the horizontal sections, such shear is shown in most of the figures on Plate IX, but the microgoniometer can obviously only be attached at points some way into the material. Not till almost the very top of the dam would anything like a parabola give a fair description of the curve of shear; a sloping straight line is nearer the mark in most cases, but something at least of the order of a quartic curve is requisite to describe even approximately the *whole* series of shear distributions.

We have already noted that this non-parabolic character of the shear curves leads to the non-linearity of the normal stresses over the horizontal sections. The first equation of (ix) shows us that

$$\widehat{xx} = \widehat{xx}_0 - \int_0^x \frac{d\widehat{zx}}{dz}\, dx \quad \text{.............................(xix).}$$

Hence by summing the difference between two shear curves for two adjacent horizontal sections, we shall find the distribution of pressures on the elements of vertical section between these horizontal sections. Fig. 9, Plate X, gives this distribution of the horizontal pressures on three sections lying respectively between G and H, H and I, I and K. For these cases, since the dam is vertical at the flank, $\widehat{xx}_0 = 0$. Two conclusions flow at once from these curves, (i) there is no approach to linearity of pressure distribution and (ii) there is no tension whatever over the vertical sections between G and K. This, of course, be it well noted, does not assert that there is no stretch along the horizontal at these sections, and it is upon this *stretch*, not tension, that rupture ultimately depends. These pressures in the dam of the Assuan type amount to considerable values, reaching on the base section to nearly 16,000 lbs. per square foot or between 7 and 8 tons per square foot.

If we take sections between which the dam is not vertical at the flank, a little consideration of the conditions of statical equilibrium shows that it is unnecessary to find $\widehat{xx}_0$ at the flank, for if d be the distance between the horizontal sections we must still have:

$$\widehat{xx} = -\frac{1}{d}\left\{\begin{array}{l}\text{area of shear curve for lower section}\\ -\text{area of shear curve for upper section}\end{array}\right\} \quad \text{...........(xx),}$$

the areas being measured up to the vertical section on which we want $\widehat{xx}$. In this manner the horizontal pressures across the vertical elements mid-way between

the horizontal sections O and N were found, and they are also given in Fig. 9, Plate X. Their maximum value does not exceed 5000 lbs. per square foot or is less than two tons per square foot.

The next stage in the work was to determine the distribution of normal stresses over a couple of horizontal sections. The sections O and G were selected for this purpose. Curves of shear on the vertical sections of the dam from GG upwards were plotted. These of course come at once from Plate IX by the fundamental property that the shears on vertical and horizontal elements at the same point are equal. Expense of reproduction prohibits the publication of the resulting $\widehat{xz}$ curves. It must suffice to state that these curves show positive shear both in substratum and dam from vertical section 10 to vertical section 23. With vertical section 24, a slight negative shear appears in the substratum, and this increases and extends, reaching to the top of the substratum but not into the dam itself, in vertical sections 27 and 28. In sections 29 to 31 all the shear has become negative, showing that not only the substratum, but an area round I, H, G in the front of the dam is subject to negative shear. This is the area of dangerous strain already referred to (see p. 16). These shear curves are not sloping straight lines, but are more nearly linear than parabolic. They were mechanically integrated from points where the vertical sections cut the horizontal sections G and O up to where they cut the surface of the dam in flank, top or front.

The second of equations (i), or the simple condition of statical equilibrium of the strip between two vertical sections, shows us that:

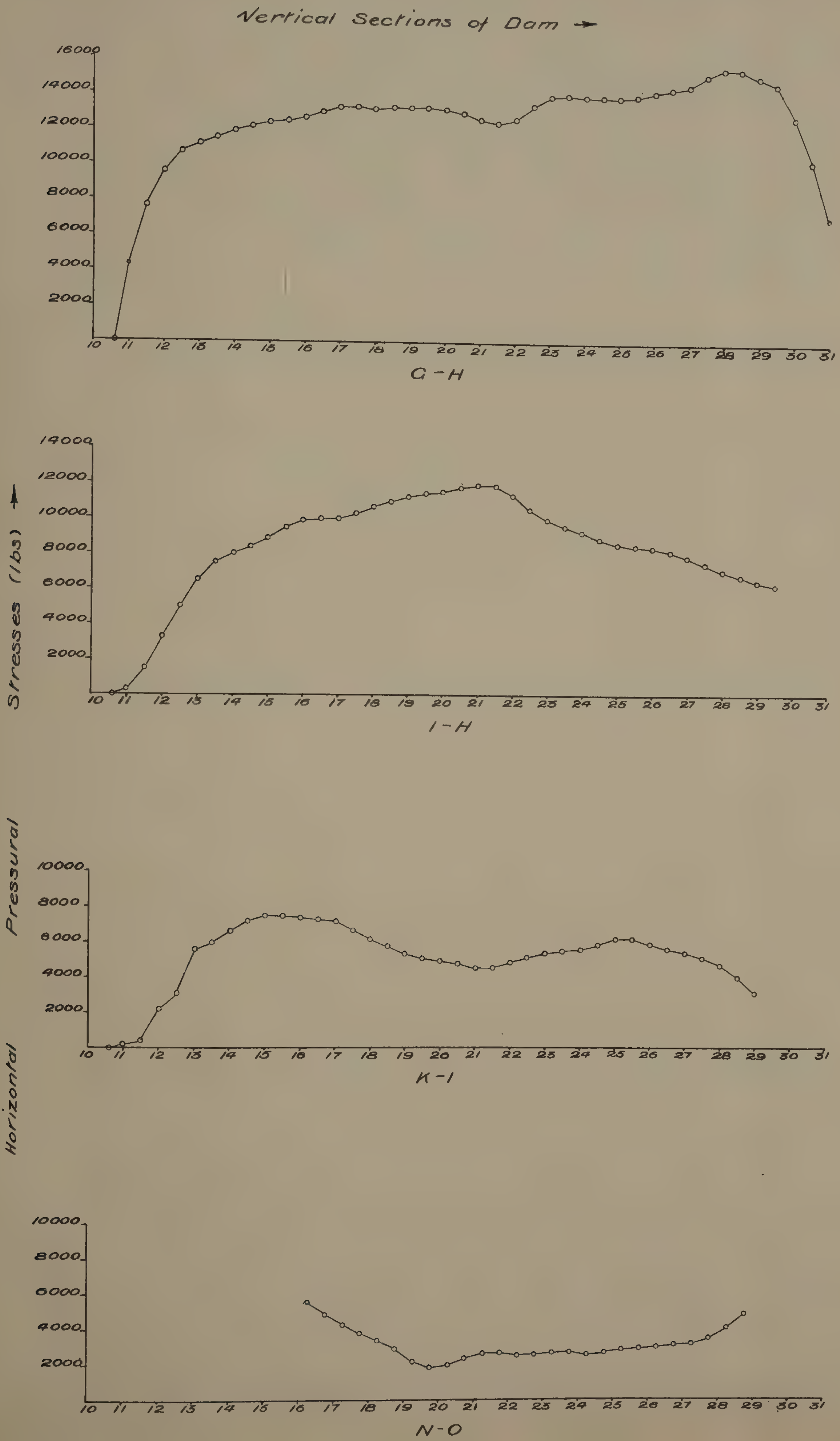
$$\widehat{zz} = -\rho g z - \frac{1}{d}(A_n - A_{n-1})^* \text{.........................(xxi)},$$

where d is the horizontal distance between two vertical sections, z is the height of masonry, and A_n and A_{n-1} are the areas of the shear curves from top of dam to the section on which $\widehat{zz}$ is measured taken for successive vertical sections.

The resulting curves of normal stress for the sections G, H, N, O, and S are given in Plate XI, Fig. 10. This plate contains the curves of pressure due to combined weight and water pressure. They differ so considerably from the shape of the dam itself—which would be their form if the dam were sliced vertically—that we realise the large influence of the vertical shear. It will be seen that for all the sections dealt with, notwithstanding the irregularity of the curves, (i) a linear distribution of vertical pressure is impossible, (ii) there exists a zone of tension towards the front of the dam, notwithstanding that the line of resistance for reservoir full falls into the 'middle-third' of the horizontal sections. This tension appears to reach its maximum about the vertical section 28 (see index-figure p. 29)

* A modification has to be made when we reach the front by the addition of $-g\rho' z'$, where ρ' is the density and z' the depth of the water.

PLATE X. ASSUAN TYPE DAM, HORIZONTAL STRESSES. FIG. 9.



and then to rise again towards the immediate front of the dam; whether it actually does so or not, is very difficult to determine accurately, because the measure of the slides upon which everything turns must always be doubtful on the actual front where the microgoniometer cannot be satisfactorily attached. The irregularities of the pressure curves are obvious and on this very account no attempt has been made throughout the work to smooth any result whatever. Every care was taken in the measurements and in the preparation of the jelly models, but the reader must remember that very slight strains had to be measured, such as could be produced by a head of water not 15 cms. at most acting on a jelly slightly more dense. No statement as to the nature of the pressure curves is made beyond the above fundamental ones, *i.e.* that they are distinctly non-linear, and that tension occurs to the fore-front of the dam, and at least half-way up it.

Finally turning to the equations for stress in terms of strain,

$$\left.\begin{aligned}\widehat{xx} &= \lambda\,(s_x + s_y + s_z) + 2\mu s_x \\ \widehat{zz} &= \lambda\,(s_x + s_y + s_z) + 2\mu s_z\end{aligned}\right\}\quad\text{(xxii)},$$

and remembering $s_y = 0$, we have

$$\left.\begin{aligned}\mu s_x &= \tfrac{1}{2}\left\{\widehat{xx} - \eta\,(\widehat{xx} + \widehat{zz})\right\} \\ \mu s_z &= \tfrac{1}{2}\left\{\widehat{zz} - \eta\,(\widehat{xx} + \widehat{zz})\right\}\end{aligned}\right\}\quad\text{(xxiii)}.$$

Hence taking $\eta = \frac{1}{4}$* we find

$$\left.\begin{aligned}\mu s_x &= \tfrac{1}{8}\,(3\widehat{xx} - \widehat{zz}) \\ \mu s_z &= \tfrac{1}{8}\,(3\widehat{zz} - \widehat{xx})\end{aligned}\right\}\quad\text{(xxiv)}.$$

These quantities are plotted for sections half-way between G and H and half-way between N and O in the accompanying diagram, Fig. 11, squeezes being measured above the axis. We see that for the Assuan type dam, there is no horizontal stretch, only squeeze under normal water pressure, it requires excessive water pressure to produce stretch. In both sections dealt with, however, there is vertical stretch towards the front of the dam. Such stretches would show themselves as horizontal cracks on the water face of the dam, low down and wherever the facing was defective. They are clearly, however, with the present water pressure, not serious, for even if μ the slide modulus were only as large as it is in sandstone—and it is likely to be much greater—the stretches are scarcely more than 1 in 50,000.

To sum up then, judging from our model dam:

(i) The present theory of stability of masonry dams is wholly erroneous. The line of resistance being inside the 'middle-third' does not prevent the material being in tension.

(ii) The distribution of vertical stresses over horizontal sections is not in

* The average for 18 kinds of stone recently investigated by Adams and Coker = ·247 and the variation is not very great either. "An Investigation into the Elastic Constants of Rocks....' *Carnegie Institute of Washington, Publication*, No. 46.

the least linear, and the distribution of shearing stress is not parabolic. There are tensions and there are stretches in the material.

(iii) A dam of the present type, however, has no tensile stress exceeding 5500 lbs. per square foot and no pressural stress exceeding 13,000 lbs. per square foot. It may, accordingly, be considered thoroughly stable in its present state.

It seems therefore desirable to discard the whole theory of the middle third and of the line of resistance, to admit that masonry does and may easily undergo tensile stress, and to fix some limit which this stress should not exceed. For example, taking a *strain* limit, perhaps a stretch of 1 in 50,000 might be considered safe.

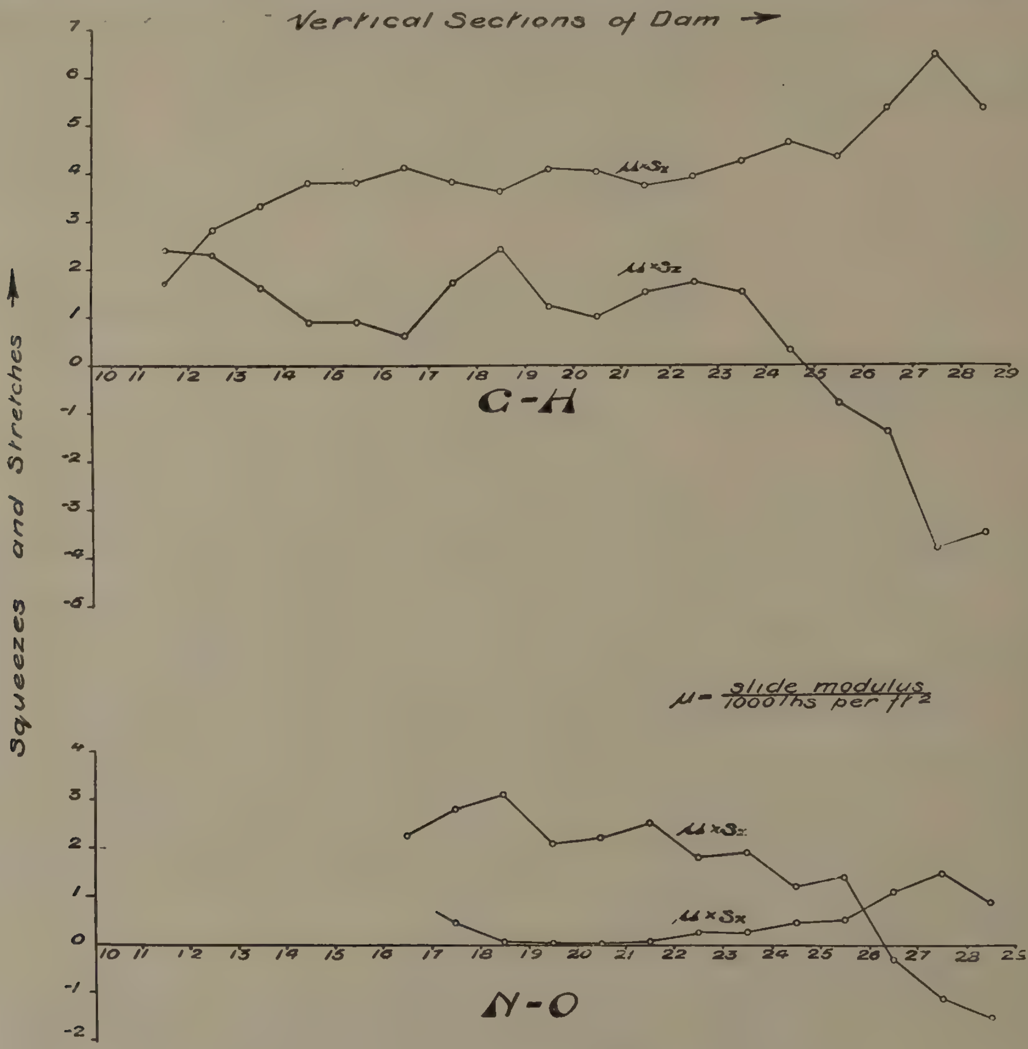


Fig. 11.

We should then reach the following processes for dealing practically within any proposed design:

(i) Form a glycerine-gelatine white pigmented jelly dam of the given contour. Determine the form and fixing of the substratum so as to represent as closely as may be feasible the local conditions. Rule the surface.

(ii) Apply water pressure and determine by the methods indicated above using either a direct or optical microgoniometer the shear distributions. Ascertain the forms of the horizontal and vertical section shear curves.

(iii) Thence by integration—of course mechanical—find the distribution of normal stresses along one or two base sections. From these deduce the stretches and squeezes and take as definite conditions of stability that the maximum stretch and squeeze shall be less than certain definite values which may be effectively fixed by experiment.

With such a test it will be found that dams like the Vyrnwy and Assuan are perfectly stable theoretically. If we apply the old theory to such dams—i.e. the absence of tension or of stretch—based upon an investigation of lines of resistance falling into the middle third, we only reach a fallacious criterion of stability. The linear distribution of normal stresses is contradicted in the most complete and absolute manner by almost any experiment on model dams.

(11) *Attempts at a semi-empirical Determination of the Stresses.*—The general result of the experiments considered in this memoir may be summed up in the words: Nothing short of a quartic curve can even *approximately* represent the slides on the horizontal sections of such dam types as the Assuan and Vyrnwy.

If the slides, however, be given approximately by a quartic, the pressure distributions over the horizontal sections will be cubical and the expression for the function V will be a quintic in the horizontal coordinate x. Let us assume:

$$V = f_0 + f_1 x + f_2 \frac{x^2}{2} + f_3 \frac{x^3}{6} + f_4 \frac{x^4}{24} + f_5 \frac{x^5}{120} \qquad \text{(xxv)},$$

where f_0, f_1, f_2, f_3, f_4, and f_5 are arbitrary functions of the vertical coordinate z. We will now endeavour to determine the forms of the f's and the values of the constants they contain. We shall not do this by trying to satisfy the surface stress conditions all over the boundary of the dam, including its base. This is absolutely impossible, because (i) the stress conditions over the base are always unknown and (ii) the solution for the differential equation for V (Eqn. (xvi) above) is not in the general case of any boundary a quintic like (xxv), but will be expressible in a doubly infinite series of terms in x and z. What we shall endeavour to do is to keep in view the experimental result of an *approximately* quartic distribution of shear, and then endeavour by using some of the boundary conditions only to obtain a general solution for the values of the f's, which might apply approximately to actual dam contours, i.e. without accurately satisfying all the boundary conditions. Such an investigation seems more likely to give serviceable results than to assume a linearity of distribution of horizontal pressures, which experiment shows to be entirely inadequate.

We have first to make V satisfy (xvi). This gives us for all values x and z:

$$f_4+f_5x+2\left(f_2''+f_3''x+f_4''\frac{x^2}{2}+f_5''\frac{x^3}{6}\right)$$
$$+f_0^{\text{iv}}+f_1^{\text{iv}}x+f_2^{\text{iv}}\frac{x^2}{2}+f_3^{\text{iv}}\frac{x^3}{6}+f_4^{\text{iv}}\frac{x^4}{24}+f_5^{\text{iv}}\frac{x^5}{120}=0 \quad \text{(xxvi)},$$

where f', f'', ..., f^{iv} are successive differentials of f with regard to z.

This leads at once to:

$$\left.\begin{array}{ll} f_4+2f_2''+f_0^{\text{iv}}=0, & \frac{1}{3}f_5''+\frac{1}{6}f_3^{\text{iv}}=0 \\ f_5+2f_3''+f_1^{\text{iv}}=0, & f_4^{\text{iv}}=0 \\ f_4''+\frac{1}{2}f_2^{\text{iv}}=0, & f_5^{\text{iv}}=0 \end{array}\right\} \quad \text{(xxvii)}.$$

Hence we must have:

$$f_4=a_0+a_1z+\tfrac{1}{2}a_2z^2+\tfrac{1}{6}a_3z^3 \quad \text{(xxviii)},$$
$$f_5=b_0+b_1z+\tfrac{1}{2}b_2z^2+\tfrac{1}{6}b_3z^3 \quad \text{(xxix)},$$

where here as later small letters are undetermined absolute constants.

From the third and fourth equations of (xxvii) we have:

$$f_2=c_0+c_1z+\tfrac{1}{2}c_2z^2+\tfrac{1}{6}c_3z^3-\tfrac{1}{12}a_2z^4-\tfrac{1}{60}a_3z^5 \quad \text{(xxx)},$$
$$f_3=d_0+d_1z+\tfrac{1}{2}d_2z^2+\tfrac{1}{6}d_3z^3-\tfrac{1}{12}b_2z^4-\tfrac{1}{60}b_3z^5 \quad \text{(xxxi)}.$$

The first two equations of (xxvii) will now give us f_0 and f_1,

$$\begin{aligned} f_0={}&g_0+g_1z+\tfrac{1}{2}g_2z^2+\tfrac{1}{6}g_3z^3 \\ &-\tfrac{1}{12}c_2z^4-\tfrac{1}{60}c_3z^5 \\ &-\tfrac{1}{24}a_0z^4-\tfrac{1}{120}a_1z^5+\tfrac{1}{240}a_2z^6+\tfrac{1}{1680}a_3z^7 \quad \text{(xxxii)}, \end{aligned}$$

$$\begin{aligned} f_1={}&h_0+h_1z+\tfrac{1}{2}h_2z^2+\tfrac{1}{6}h_3z^3 \\ &-\tfrac{1}{12}d_2z^4-\tfrac{1}{60}d_3z^5 \\ &-\tfrac{1}{24}b_0z^4-\tfrac{1}{120}b_1z^5+\tfrac{1}{240}b_2z^6+\tfrac{1}{1680}b_3z^7 \quad \text{(xxxiii)}. \end{aligned}$$

We have thus found the value of V in terms of 24 arbitrary absolute constants. We have at once from (xv):

$$\widehat{zz}=f_2+f_3x+\tfrac{1}{2}f_4x^2+\tfrac{1}{6}f_5x^3-g\rho z \quad \text{(xxxiv)},$$

an equation giving the cubical distribution of pressure.

$$\widehat{xx}=f_0''+f_1''x+\tfrac{1}{2}f_2''x^2+\tfrac{1}{6}f_3''x^3+\tfrac{1}{24}f_4''x^4+\tfrac{1}{120}f_5''x^5 \quad \text{(xxxv)},$$

an equation showing that the pressure on the vertical sections, i.e. $x=$constant, will be quintic in distribution.

$$\widehat{xz}=-(f_1'+f_2'x+\tfrac{1}{2}f_3'x^2+\tfrac{1}{6}f_4'x^3+\tfrac{1}{24}f_5'x^4) \quad \text{(xxxvi)},$$

an equation showing that the shear is quartic on the horizontal and sextic on the vertical sections.

Now apart from the unknown stresses on the base we have the following conditions, supposing the origin at the meet of the front and top, and the batter of the front, or angle between axis of z and front to be β, $(\tan\beta=\epsilon)$:

(A) At the top or $z=0$,

$$\widehat{zz}=0,\quad \widehat{xz}=0,\quad \text{for all values of } x \text{ ....(xxxvii).}$$

(B) At the front, $x=-z\tan\beta$,

$$\left.\begin{aligned}-p\cos\beta&=\widehat{xx}\cos\beta+\widehat{xz}\sin\beta\\ -p\sin\beta&=\widehat{xz}\cos\beta+\widehat{zz}\sin\beta\end{aligned}\right\}\text{.....................(xxxviii).}$$

Or

$$\left.\begin{aligned}-g\rho' z&=\widehat{xx}+\widehat{xz}\epsilon\\ -g\rho' z\epsilon&=\widehat{xz}+\widehat{zz}\epsilon\end{aligned}\right\}\text{ ....(xxxix),}$$

where ρ' is the density of water, for all values of z, subject to the relation $x=-z\tan\beta$.

(C) The shear and pressure at each element of surface of the flank is to be zero.

Now to what extent can we satisfy (A), (B) and (C)? Suppose we attempt to make the condition (A) accurately satisfied, then we must have

$$c_0+d_0x+\tfrac{1}{2}a_0x^2+\tfrac{1}{6}b_0x^3=0,$$
$$h_1+c_1x+\tfrac{1}{2}d_1x^2+\tfrac{1}{6}a_1x^3+\tfrac{1}{24}b_1x^4=0$$

for all values of x. Therefore the nine arbitrary constants involved must all be zero. Thus to satisfy accurately the stress conditions at the top of the dam costs us 9 out of our available 24 constants.

Now consider the values for the stresses, equations (xxxiv)—(xxxvi). It is clear that the constants h_0, g_0, g_1 do not appear in them; in other words 12 of the 24 constants are determined. We shall find that it requires 12 more constants to satisfy the conditions at the front of the dam. *In other words no quartic would suffice to satisfy accurately more than the conditions at top and front of any dam. It would leave the flank and base conditions unsatisfied.*

The satisfaction of top and flank conditions only leads us to a solution for indefinitely extended masonry, ice and rock masses, which, however, as having more interest for geological problems, has been treated in a separate paper.

Next let us suppose the distribution of pressure over a horizontal section to be linear. All we have to do is to put f_4 and f_5 zero in equations (xxxiv)—(xxxvi), or we have a_0, a_1, a_2, a_3, b_0, b_1, b_2, b_3 all zero. Suppose $p_0=0$, we find:

$$\widehat{zz}=c_0+c_1z+\frac{2\epsilon^3d_1}{1+3\epsilon^2}z^2+(d_0+d_1z)\,x-g\rho z\text{ ....(xl),}$$

$$\begin{aligned}\widehat{xx}=\epsilon^2c_0-(g\rho'+2\epsilon^2g(\rho-\rho')-3\epsilon^2c_1+2\epsilon^3d_0)\,z-\frac{4\epsilon^3d_1}{1+3\epsilon^2}z^2\\ -\left(g(\rho-\rho')\,\epsilon-2\epsilon c_1+\epsilon^2d_0+\frac{3\epsilon^2(1-\epsilon^2)\,d_1}{1+3\epsilon^2}z\right)x+\frac{2\epsilon^3d_1}{1+3\epsilon^2}x^2\text{ ....(xli),}\end{aligned}$$

$$\widehat{xz} = -\left\{\epsilon c_0 - (g(\rho - \rho') - 2\epsilon c_1 + \epsilon^2 d_0) z - \frac{3}{2}\frac{\epsilon^2(1-\epsilon^2) d_1}{1+3\epsilon^2} z^2 + \left(c_1 + \frac{4\epsilon^3 d_1}{1+3\epsilon^2} z\right) x + \tfrac{1}{2} d_1 x^2\right\} \quad \text{(xlii)}.$$

We have here only four constants: c_0, c_1, d_0, d_1, to be determined by the two series of conditions at the top and flank of the dam; further we have really assumed the shear over the base to be parabolic, which experimentally it is not. Since ϵ is small in actual dams, ϵ^3 is very small, and the z^2 term in $\widehat{xx}$ is negligible. We have then the results of the first memoir in which $\widehat{zz}$ was taken as linear, $\widehat{xx}$ as linear and $\widehat{xz}$ as parabolic. If we wish to accurately satisfy the conditions at the top we must have $\widehat{xz}$ and $\widehat{zz} = 0$ for $z = 0$ whatever be x. This involves $c_0 = 0$, $d_0 = 0$, $c_1 = 0$, and $d_1 = 0$, or the solution reduces to that referred to p. 37 and discussed in a separate paper. We can satisfy no flank conditions and no base conditions. No finite number of constants will as a general rule admit of our accurately satisfying all the surface conditions. We can only make the resultant stress across any section equal to the applied load above that section. This involves: (i) resultant normal load = resultant pressure in magnitude, (ii) be opposite to it in direction, and (iii) the resultant shearing load must equal the resultant shearing stress. If we are dealing with the top with no resultant loads (ii) has to be disregarded. Accordingly it takes three constants to satisfy the equivalence of resultant loads and stresses over any horizontal section except for the top, where two will suffice. In other words we cannot even approximately satisfy the equivalence of resultant load and stress at top and base by assuming linearity of pressure on the horizontal sections. It would require five constants and we have only four available. If we take the quartic distribution of shear we have eight available constants, and we can accordingly satisfy the equivalence of resultant load and stress at top and base, this will consume five constants. We are left with three constants and we can expend these in making an equivalence of resultant load at the mid section of the dam. In neither case shall we have accurately satisfied the vanishing of the stresses at the flank, but in the former case we shall only have insured that the resultant stresses on the flank *from top to bottom* form a system with a zero resultant; in the latter case the resultant stresses on the upper and on the lower halves of the flank form each independently systems in equilibrium and we shall accordingly have a nearer approximation—although of course only rough—to the condition of affairs in an actual dam in which the shearing and normal stresses throughout the whole length of the flank are zero.

We can now sum up these three dam theories.

A. Old Middle Third Theory. This assumes linearity of pressures on horizontal sections, and determines the pressures by equivalence of resultant stress at every horizontal section with load. This theory does not satisfy the surface conditions at front, flank or top.

B. Mathematical Theory of linearity of pressure on horizontal sections. This satisfies absolutely the surface stress conditions at the front; makes the shear on any horizontal section parabolic, and can be made to satisfy the equivalence of load and stress at base fully, but it will not satisfy it completely at any other section. It only satisfies the flank and top conditions, in so far as it makes the total stress over them zero. It fails to give the shear on the horizontal sections a form in the least like that observed in experiment.

C. Theory of Quartic Distribution of Shear. This gives a general shape to the shear curve approximately accordant with experiment. It satisfies the internal equations of elasticity; it completely satisfies the stress conditions at the front; it can have its constants selected to give the equivalence of resultant stress and applied load at top, mid-horizontal section and base. It involves a distribution of stresses over the flank, which ought to be zero, but is only such that the stresses have a zero resultant for the upper and lower halves of the flank. B, however, only provides for the vanishing of the resultant of the stresses over the entire flank. Theory A did not deal with the shearing stress at all, but while it satisfied the equivalence of resultant pressure and load at each horizontal section on the assumption of linearity of normal pressure, obtained results, which were quite inconsistent with the general equations of elasticity except for very special forms of flank.

We now proceed to determine the constants of the approximate solution C on the following assumptions:

(i) Quartic distribution of shear on the horizontal sections.

(ii) Conditions at front absolutely fulfilled.

(iii) Resultant load (pressural and shearing) equivalent to resultant stress at top, mid-horizontal section, and base.

(iv) We shall suppose—as is the case in most actual dams—that ϵ the batter is so small that its square may be neglected. We shall further neglect the atmospheric pressure.

Equations (xxxviii) and (xxxix) provide us at once with:

$$a_2=0,\quad a_3=0,\quad b_2=0,\quad b_3=0,\quad c_2=-\tfrac{1}{2}a_0,\quad c_3=-\tfrac{1}{2}a_1,$$

$$d_2=\epsilon a_1-\tfrac{1}{2}b_0,\quad d_3=-\tfrac{1}{2}b_1,\quad h_1=\epsilon c_0,\quad h_2=-g(\rho-\rho')\epsilon+2\epsilon c_1,\quad h_3=-\tfrac{3}{2}\epsilon a_0,$$

$$g_2=0,\quad g_3=-g\rho'.$$

Hence for the functions:

$$\begin{aligned}
f_0'' &= -g\rho' z,\\
f_1' &= \epsilon c_0-\{g(\rho-\rho')\epsilon-2\epsilon c_1\}z-\tfrac{3}{4}\epsilon a_0 z^2-\tfrac{1}{3}\epsilon a_1 z^3,\\
f_2 &= c_0+c_1 z-\tfrac{1}{4}a_0 z^2-\tfrac{1}{12}a_1 z^3,\\
f_3 &= d_0+d_1 z+(\tfrac{1}{2}\epsilon a_1-\tfrac{1}{4}b_0)z^2-\tfrac{1}{12}b_1 z^3,\\
f_4 &= a_0+a_1 z,\qquad f_5=b_0+b_1 z.
\end{aligned}$$

whence we find for the stresses:

$$\widehat{zz} = c_0 + (c_1 - g\rho)\,z - \tfrac{1}{4}a_0 z^2 - \tfrac{1}{12}a_1 z^3 + \{d_0 + d_1 z + (\tfrac{1}{2}\epsilon a_1 - \tfrac{1}{4}b_0)\,z^2 - \tfrac{1}{12}b_1 z^3\}\,x$$
$$+ (\tfrac{1}{2}a_0 + \tfrac{1}{2}a_1 z)\,x^2 + (\tfrac{1}{6}b_0 + \tfrac{1}{6}b_1 z)\,x^3 \qquad \text{(xliii)}.$$

$$\widehat{xx} = -g\rho' z + [-\{g(\rho - \rho')\,\epsilon - 2\epsilon c_1\} - \tfrac{3}{2}\epsilon a_0 z - \epsilon a_1 z^2]\,x + \{-\tfrac{1}{4}a_0 - \tfrac{1}{4}a_1 z\}\,x^2$$
$$+ \{(\tfrac{1}{6}\epsilon a_1 - \tfrac{1}{12}b_0) - \tfrac{1}{12}b_1 z\}\,x^3 \qquad \text{(xliv)}.$$

$$\widehat{xz} = -[\epsilon c_0 - \{g(\rho - \rho')\,\epsilon - 2\epsilon c_1\}\,z - \tfrac{3}{4}\epsilon a_0 z^2 - \tfrac{1}{3}\epsilon a_1 z^3 + \{c_1 - \tfrac{1}{2}a_0 z - \tfrac{1}{4}a_1 z^2\}\,x$$
$$+ \{\tfrac{1}{2}d_1 + (\tfrac{1}{2}\epsilon a_1 - \tfrac{1}{4}b_0)\,z - \tfrac{1}{8}b_1 z^2\}\,x^2 + \tfrac{1}{6}a_1 x^3 + \tfrac{1}{24}b_1 x^4] \qquad \text{(xlv)}.$$

We now require to find the integral shears and pressures and the moment of the latter over any horizontal section. Consider a section at depth α of breadth $\beta + \alpha\epsilon$, and let V be the resultant vertical pressure due to water and weight of masonry over this section and M the moment of V about the point in which the axis of z meets the section, then

$$V = -\int_{-\alpha\epsilon}^{\beta} \widehat{zz}\,dx \qquad \text{(xlvi)},$$

$$M = -\int_{-\alpha\epsilon}^{\beta} x\widehat{zz}\,dx \qquad \text{(xlvii)}.$$

Further if S be the resultant shear due to the water pressure above the section,

$$S = -\int_{-\alpha\epsilon}^{\beta} \widehat{xz}\,dx \qquad \text{(xlviii)}.$$

We shall now find V, M and S, remembering that ϵ^2 is to be neglected. We shall further rearrange the terms under the constants. We find if $\gamma = \beta + \alpha\epsilon$,

$$g\rho\alpha\,(\beta + \alpha\epsilon) - V = a_0\,(\tfrac{1}{6}\beta^3 - \tfrac{1}{4}\alpha^2\gamma) + a_1\,(\tfrac{1}{6}\alpha\beta^3 + \tfrac{1}{4}\epsilon\alpha^2\beta^2 - \tfrac{1}{12}\alpha^3\gamma)$$
$$+ b_0\,(\tfrac{1}{24}\beta^4 - \tfrac{1}{8}\beta^2\alpha^2) + b_1\,(\tfrac{1}{24}\alpha\beta^4 - \tfrac{1}{24}\alpha^3\beta^2)$$
$$+ c_0\gamma + c_1\alpha\gamma + d_0\tfrac{1}{2}\beta^2 + d_1\tfrac{1}{2}\alpha\beta^2 \qquad \text{(xlix)},$$

$$\tfrac{1}{2}g\rho\alpha\beta^2 - M = a_0\,(\tfrac{1}{8}\beta^4 - \tfrac{1}{8}\alpha^2\beta^2) + a_1\,(\tfrac{1}{8}\alpha\beta^4 + \tfrac{1}{6}\epsilon\alpha^2\beta^3 - \tfrac{1}{24}\alpha^3\beta^2)$$
$$+ b_0\,(\tfrac{1}{30}\beta^5 - \tfrac{1}{12}\alpha^2\beta^3) + b_1\,(\tfrac{1}{30}\alpha\beta^5 - \tfrac{1}{36}\alpha^3\beta^3)$$
$$+ c_0\tfrac{1}{2}\beta^2 + c_1\tfrac{1}{2}\alpha\beta^2 + d_0\tfrac{1}{3}\beta^3 + d_1\tfrac{1}{3}\alpha\beta^3 \qquad \text{(l)},$$

$$g(\rho - \rho')\,\epsilon\alpha\beta + S = a_0\,(-\tfrac{1}{4}\alpha\beta^2 - \tfrac{3}{4}\epsilon\alpha^2\beta) + a_1\,(\tfrac{1}{24}\beta^4 + \tfrac{1}{6}\epsilon\alpha\beta^3 - \tfrac{1}{8}\alpha^2\beta^2 - \tfrac{1}{3}\epsilon\alpha^3\beta)$$
$$+ b_0\,(-\tfrac{1}{12}\alpha\beta^2) + b_1\,(\tfrac{1}{120}\beta^5 - \tfrac{1}{24}\alpha^2\beta^3)$$
$$+ c_0\epsilon\beta + c_1\,(2\epsilon\alpha\beta + \tfrac{1}{2}\beta^2) + d_0 \times 0 + d_1\tfrac{1}{6}\beta^2 \qquad \text{(li)}.$$

Now at the top V and $S = 0$ for $\alpha = 0$. Hence we have from the first and last of these equations, if β_0 = length of top:

$$0 = a_0\tfrac{1}{6}\beta_0^3 + b_0\tfrac{1}{24}\beta_0^4 + c_0\beta_0 + d_0\tfrac{1}{2}\beta_0^2 \qquad \text{(lii)},$$

$$0 = a_1\tfrac{1}{24}\beta_0^4 + b_1\tfrac{1}{120}\beta_0^5 + c_0\epsilon\beta_0 + c_1\tfrac{1}{2}\beta_0^2 + d_1\tfrac{1}{6}\beta_0^3 \qquad \text{(liii)}.$$

Now let V_1, S_1, M_1 be the values at the mid-section, where we will say $\alpha = \alpha_1$, $\beta = \beta_1$, and V_2, S_2, M_2 the values at the base, where $\alpha = \alpha_2$, $\beta = \beta_2$, then (xlix) to

(li) provide six further equations, which with (lii) and (liii) suffice to determine the eight constants.

It was desirable to illustrate the whole process by numerical calculation* in the case of an actual dam—say, the dam of the Assuan type dealt with in this paper. In this case:

$$\alpha_2 = 100', \quad \alpha_1 = 50', \quad \beta_0 = 10', \quad \beta_1 = 29'{\cdot}83, \quad \beta_2 = 67'{\cdot}17,$$
$$\epsilon = 1/16, \quad \gamma_1 = 32'{\cdot}96, \quad \gamma_2 = 73'{\cdot}42.$$

If we take our forces over unit breadth of dam and measure them in cubic feet of masonry we shall have:

$$g\rho\alpha_1(\beta_1 + \alpha_1\epsilon) = 1648 \text{ cubic feet of masonry},$$
$$g\rho\alpha_2(\beta_2 + \alpha_2\epsilon) = 7342 \quad ,, \quad ,, \quad ,,$$
$$g(\rho - \rho')\,\epsilon\alpha_1\gamma_1 = 57{\cdot}22 \quad ,, \quad ,, \quad ,,$$
$$g(\rho - \rho')\,\epsilon\alpha_2\gamma_2 = 256{\cdot}04 \quad ,, \quad ,, \quad ,,$$
$$\tfrac{1}{2}g\rho\alpha_1\beta_1^2 = 22245{\cdot}72 \text{ (feet)}^4 \text{ of masonry},$$
$$\tfrac{1}{2}g\rho\alpha_2\beta_2^2 = 225590{\cdot}44 \quad ,, \quad ,, \quad ,,$$
$$V_1 = 1017{\cdot}9, \quad S_1 = 553{\cdot}6, \quad M_1 = 12703\dagger,$$
$$V_2 = 3735{\cdot}7, \quad S_2 = 2228{\cdot}6, \quad M_2 = 140948,$$

all in units of cubic feet of masonry and feet.

If these values be substituted in the eight equations and the constants found, we shall obtain, lengths being measured in feet, the stresses in feet of masonry. The eight equations are:

$$\cdot 04\bar{a}_0 + \cdot 001\bar{b}_0 + 2{\cdot}4\bar{c}_0 + 1{\cdot}2\bar{d}_0 = 0,$$
$$\cdot 1\bar{a}_1 + \cdot 002\bar{b}_1 + 1{\cdot}5\bar{c}_0 + 1{\cdot}2\bar{c}_1 + \cdot 4\bar{d}_1 = 0,$$

$$\left.\begin{array}{l} -1{\cdot}617607\bar{a}_0 - \cdot 873778\bar{a}_1 - \cdot 245080\bar{b}_0 - \cdot 298495\bar{b}_1 \\ +3{\cdot}296\bar{c}_0 + 1{\cdot}648\bar{c}_1 + 4{\cdot}449145\bar{d}_0 + 2{\cdot}224572\bar{d}_1 \end{array}\right\} = 6{\cdot}301,$$

$$\left.\begin{array}{l} -1{\cdot}461856\bar{a}_0 - 3{\cdot}089376\bar{a}_1 - \cdot 110598\bar{b}_0 - \cdot 256813\bar{b}_1 \\ + \cdot 186438\bar{c}_0 + \cdot 631352\bar{c}_1 + \cdot 442383\bar{d}_1 \end{array}\right\} = 6{\cdot}1082,$$

$$\left.\begin{array}{l} - \cdot 600393\bar{a}_0 + \cdot 337055\bar{a}_1 - \cdot 158988\bar{b}_0 - \cdot 177002\bar{b}_1 \\ +1{\cdot}4915\bar{c}_0 + \cdot 74575\bar{c}_1 + 2{\cdot}966096\bar{d}_0 + 1{\cdot}483048\bar{d}_1 \end{array}\right\} = 3{\cdot}199035,$$

$$\left.\begin{array}{l} -1{\cdot}330403\bar{a}_0 - \cdot 362393\bar{a}_1 - \cdot 479158\bar{b}_0 - 1{\cdot}031736\bar{b}_1 \\ + \cdot 7342\bar{c}_0 + \cdot 7342\bar{c}_1 + 2{\cdot}255904\bar{d}_0 + 2{\cdot}255904\bar{d}_1 \end{array}\right\} = 3{\cdot}6063,$$

$$\left.\begin{array}{l} -1{\cdot}442812\bar{a}_0 - 5{\cdot}875266\bar{a}_1 - \cdot 252549\bar{b}_0 - 1{\cdot}148797\bar{b}_1 \\ + \cdot 041981\bar{c}_0 + \cdot 309553\bar{c}_1 + \cdot 505097\bar{d}_1 \end{array}\right\} = 2{\cdot}48464,$$

$$\left.\begin{array}{l} - \cdot 460802\bar{a}_0 + 1{\cdot}459458\bar{a}_1 - \cdot 308129\bar{b}_0 - \cdot 574733\bar{b}_1 \\ + \cdot 33585\bar{c}_0 + \cdot 33585\bar{c}_1 + 1{\cdot}503936\bar{d}_0 + 1{\cdot}503936\bar{d}_1 \end{array}\right\} = 1{\cdot}2601,$$

* The whole of the numerical work was carried out by Mr John Blakeman, M.Sc.

† Distance of load-point from axis of z for mid-section $= 12'{\cdot}48$ and for base $= 37'{\cdot}73$.

where : $\bar{a}_0 = 100a_0, \quad \bar{a}_1 = 1000a_1, \quad \bar{b}_0 = 10000b_0, \quad \bar{b}_1 = 100000b_1,$

$$\bar{c}_0 = \tfrac{1}{10}c_0, \quad \bar{c}_1 = 10c_1, \quad \bar{d}_0 = d_0, \quad \bar{d}_1 = 100d_1.$$

The solution of these equations was:

$$\begin{aligned} \bar{a}_0 &= -\ 60{\cdot}685857, & \bar{a}_1 &= 11{\cdot}200565, \\ \bar{b}_0 &= 1060{\cdot}938555, & \bar{b}_1 &\ -155{\cdot}463084, \\ \bar{c}_0 &= 26{\cdot}570147, & \bar{c}_1 &= -\ 99{\cdot}780670, \\ \bar{d}_0 &= 52{\cdot}001214, & \bar{d}_1 &= 197{\cdot}681133. \end{aligned}$$

Hence determining the shear $\widehat{xz}$ in feet of masonry:

$$\begin{aligned} \widehat{xz} = &-16{\cdot}606342 + 1{\cdot}281981z - {\cdot}028447z^2 \\ &+ {\cdot}000233z^3 + 9{\cdot}978067x - {\cdot}303429xz \\ &+ {\cdot}002800xz^2 - {\cdot}988405x^2 + {\cdot}026173x^2z \\ &- {\cdot}000194x^2z^2 - {\cdot}001867x^3 + {\cdot}000065x^4. \end{aligned}$$

Now this shear has been determined so that load and resultant stress balance at top, mid and basal sections of an actual dam, and that the surface conditions are accurately satisfied at the front. The flank has been disregarded. It must clearly, however, have zero resultant stress between top and mid sections and between mid and basal sections. The curves of shear were then plotted from top to basal sections at intervals of 10 feet. What was the result when these curves were compared with the experimental values given in Plate IX?

The resulting distribution on the basal section showed indeed a  shaped curve with negative shear to the front and positive to the flank, but the point of zero shear was very different from that of the experiments. The mid-section also showed positive and negative shear, differing thus *in toto* from the experimental mid-section. All the other sections (top of course excluded) practically gave total shears only partially balanced by the water pressure. They thus proved that our assumptions lead to a very sensible series of *unbalanced stresses over the flank*, only balanced as a whole for the upper and lower divisions of the dam.

Other attempts with various other assumptions were made—equally unsuccessfully. If the reader ask why this large expenditure of energy on very laborious arithmetic, the answer must be: It sufficed to show that no solution which merely satisfies resultant load and stress equivalence at certain arbitrary sections is at all likely to be even roughly satisfactory. Whatever be the nature of the accurate or even of an approximate solution, it must pay as much attention to the absence of stress on the flank as to the presence of pressure on the front. Any

theoretical assumptions of linear, parabolic or even quartic distributions of shear so far fail to achieve this. There seems very little hope on such lines of an approximate theoretical solution of the problem. Solutions of Eqn. (xvi) in conjugate functions, roughly approaching dam contour forms, might be suggestive; possibly a graphical method may ultimately be devised of interpreting (xv) and (xvi). But the conclusion to which we at present incline is that the stability of dams should for the time be tested by an empirical-graphical method, i.e., the experiment on the model dam followed by the graphical deduction of the stresses and strains from the slide curves in the manner indicated in this memoir. The current theory is wholly inapplicable, and merely serves to screen the purely empirical design by a fallacious appearance of theoretical justification. The stresses given by the current theory will certainly not necessarily be within 50 per cent. of the values of the stresses in a theoretical homogeneous dam of isotropic material of the same form; it probably would be no exaggeration to say within 100 per cent. of the values of the stresses in the actual dam. We think it possible to reach, by the model dam, stresses within 10 per cent. of those in the theoretical dam, and possibly within 30 to 40 per cent. of those in the actual dam. We do not believe that if a complete mathematical theory existed, it would really give a better result; for, everything depends on the basal distribution of shear, and this will be largely influenced, not only by the local conditions of the substratum, but also by the manner in which the dam is set up, and the exact shape of the front and flank at the union with the substratum, which it will usually be impossible to determine *a priori*.

List of Plates and Figures.

Plates.

* By an error of the engraver's printer, Plate I, Fig. (i), and Plate III, Fig. (i), are printed as above, but the reference on p. 22 to Plate III, Fig. (i), should then be to Plate I, Fig. (i).

Figures.

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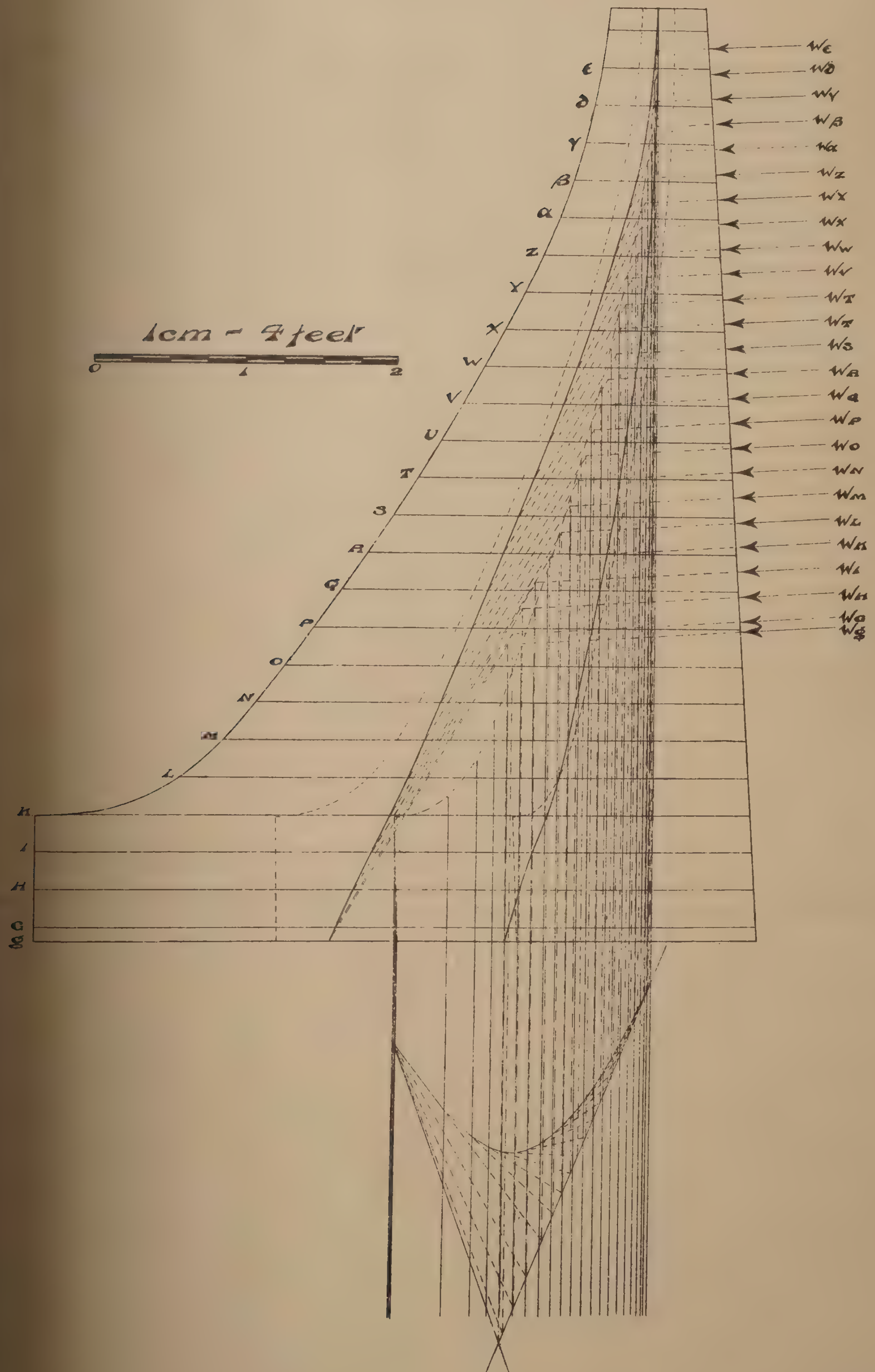
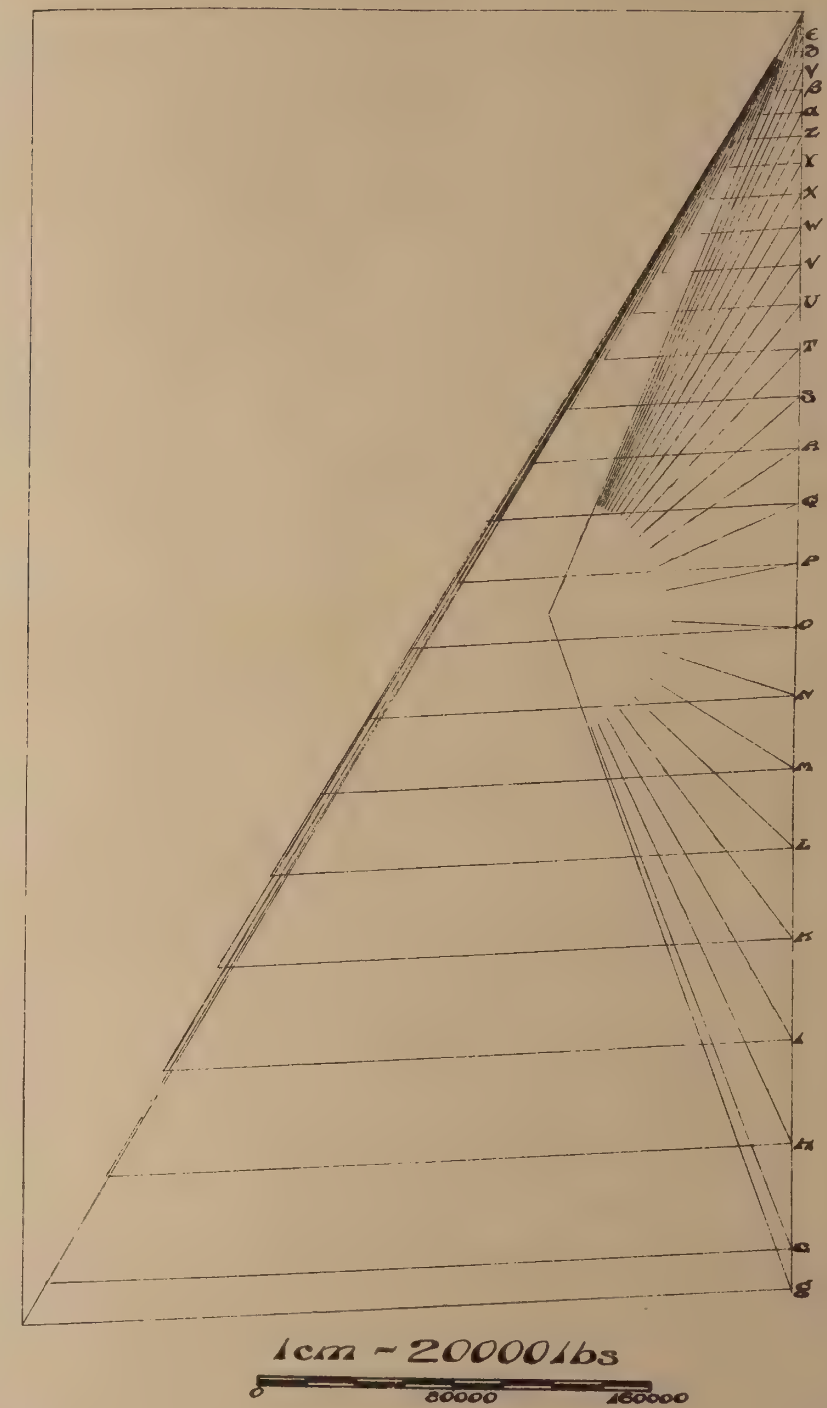
LINES OF RESISTANCE ASSUAN TYPE DAM

PLATE VII





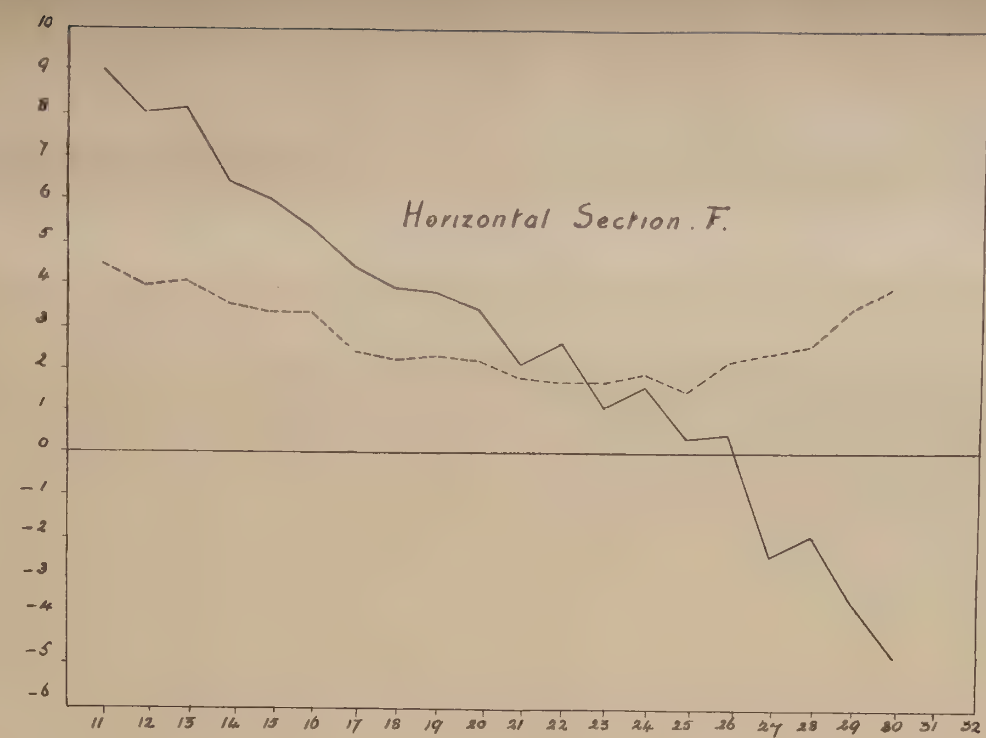
Diagrams of Shear

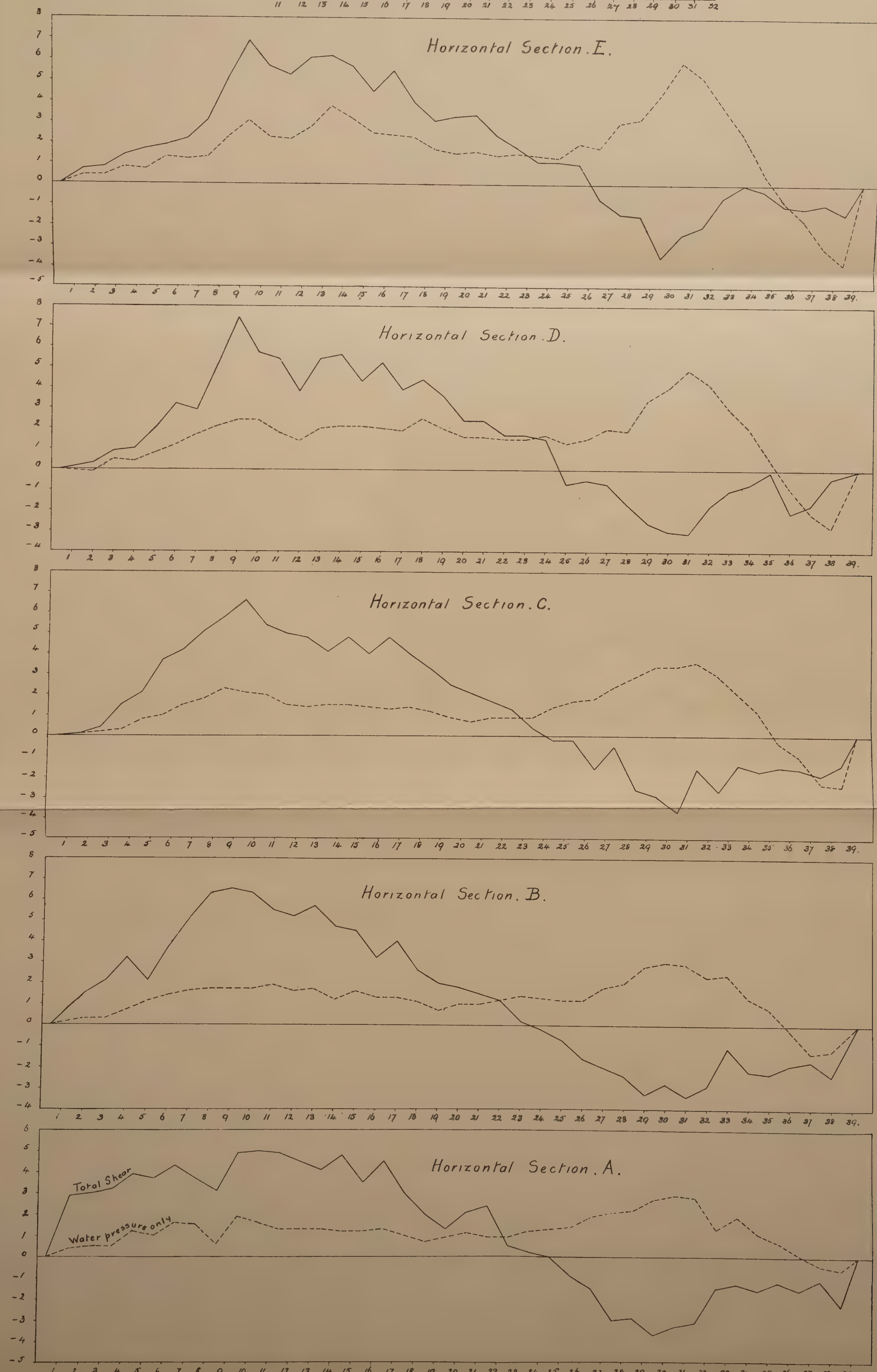
Full Lines = Total Shear

Dotted Lines = Water pressure only

Vertical Scale = 1000 lbs per sq foot.

Horizontal Scale: see Index figure to Dam





Diagrams of Shear

Vertical Scale = 1000 lbs per sq foot

Horizontal Scale: see Index figure to Dam.

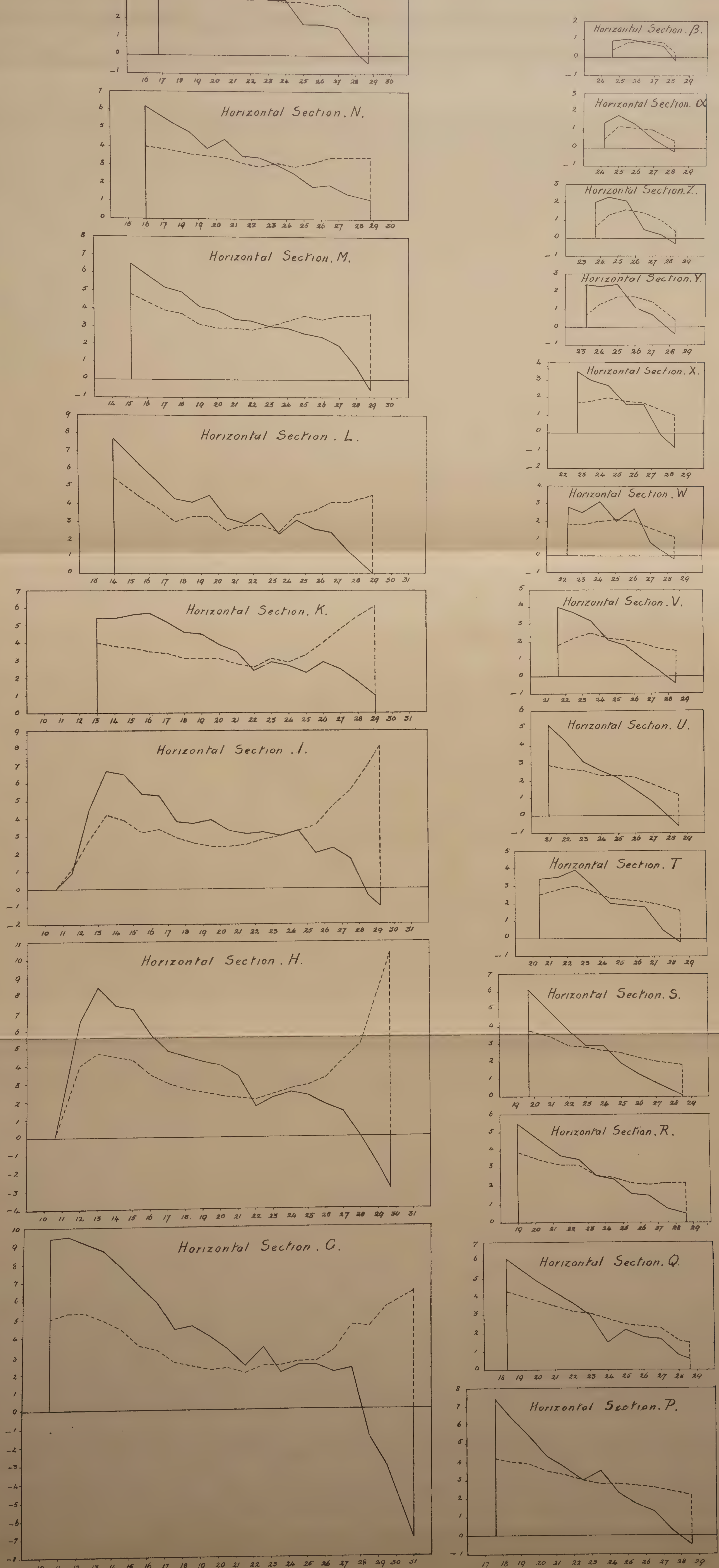
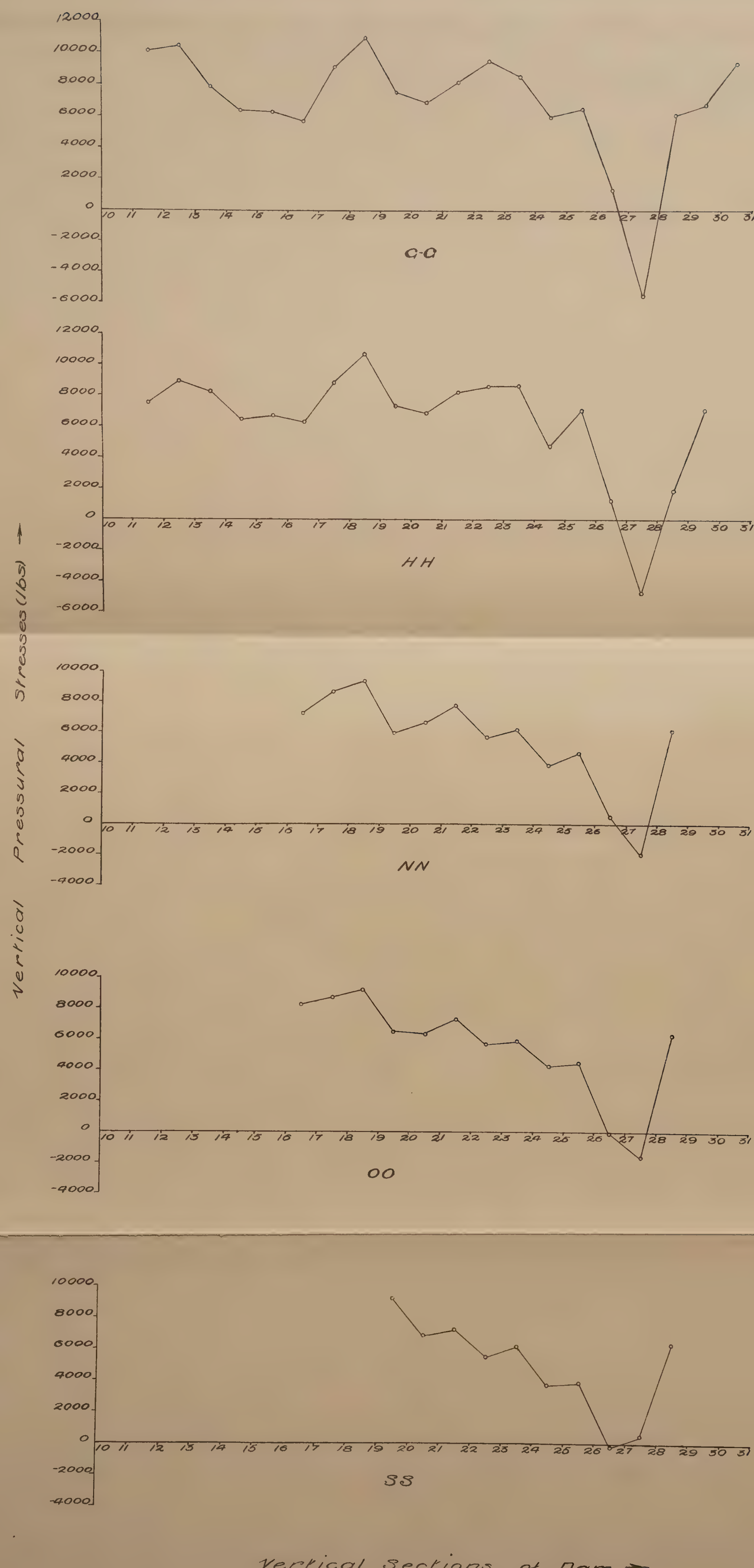


PLATE XI. FIG. 10. ASSUAN TYPE DAM.



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